ELE 273- Homework #2

Problem 8. You enter a special kind of chess tournament, in which you play one game with each of three opponents, but you get to choose the order in which you play your opponents, knowing the probability of a win against each. You win the tournament if you win two games in a row, and you want to maximize the probability of winning. Show that it is optimal to play the weakest opponent second, and that the order of playing the other two opponents does not matter.

Problem 16. We are given three coins: one has heads in both faces, the second has tails in both faces, and the third has a head in one face and a tail in the other. We choose a coin at random, toss it, and the result is heads. What is the probability that the opposite face is tails?

Problem 18. Let A and B be events. Show that $\mathbf{P}(A \cap B \mid B) = \mathbf{P}(A \mid B)$, assuming that $\mathbf{P}(B) > 0$.

Problem 23. We have two jars, each initially containing an equal number of balls. We perform four successive ball exchanges. In each exchange, we pick simultaneously and at random a ball from each jar and move it to the other jar. What is the probability that at the end of the four exchanges all the balls will be in the jar where they started?

Problem 27. Alice and Bob have $2n + 1$ coins, each coin with probability of heads equal to $1/2$. Bob tosses $n+1$ coins, while Alice tosses the remaining n coins. Assuming independent coin tosses, show that the probability that after all coins have been tossed, Bob will have gotten more heads than Alice is $1/2$.

Problem 30. A hunter has two hunting dogs. One day, on the trail of some animal, the hunter comes to a place where the road diverges into two paths. He knows that each dog, independent of the other, will choose the correct path with probability p. The hunter decides to let each dog choose a path, and if they agree, take that one, and if they disagree, to randomly pick a path. Is his strategy better than just letting one of the two dogs decide on a path?

Problem 39. A particular class has had a history of low attendance. The annoyed professor decides that she will not lecture unless at least k of the n students enrolled in the class are present. Each student will independently show up with probability p_g if the weather is good, and with probability p_b if the weather is bad. Given the probability of bad weather on a given day, obtain an expression for the probability that the professor will teach her class on that day.

Problem 53. Ninety students, including Joe and Jane, are to be split into three classes of equal size, and this is to be done at random. What is the probability that Joe and Jane end up in the same class?

Problem 57. How many 6-word sentences can be made using each of the 26 letters of the alphabet exactly once? A word is defined as a nonempty (possibly jibberish) sequence of letters.

Problem 59. A parking lot contains 100 cars, k of which happen to be lemons. We select m of these cars at random and take them for a test drive. Find the probability that n of the cars tested turn out to be lemons.

Problem 4. An internet service provider uses 50 modems to serve the needs of 1000 customers. It is estimated that at a given time, each customer will need a connection with probability 0.01, independent of the other customers.

(a) What is the PMF of the number of modems in use at the given time?

(b) Repeat part (a) by approximating the PMF of the number of customers that need a connection with a Poisson PMF.

(c) What is the probability that there are more customers needing a connection than there are modems? Provide an exact, as well as an approximate formula based on the Poisson approximation of part (b).

Problem 16. Let X be a random variable with PMF

$$p_X(x) = \begin{cases} x^2/a, & \text{if } x = -3, -2, -1, 0, 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find a and $\mathbf{E}[X]$.

(b) What is the PMF of the random variable $Z = \big(X - \mathbf{E}[X]\big)^2$?

(c) Using the result from part (b), find the variance of X.

(d) Find the variance of X using the formula $\text{var}(X) = \sum_x \big(x - \mathbf{E}[X]\big)^2 p_X(x)$.

Problem 17. A city's temperature is modeled as a random variable with mean and standard deviation both equal to 10 degrees Celsius. A day is described as "normal" if the temperature during that day ranges within one standard deviation from the mean. What would be the temperature range for a normal day if temperature were expressed in degrees Fahrenheit?

Problem 22. Two coins are simultaneously tossed until one of them comes up a head and the other a tail. The first coin comes up a head with probability p and the second with probability q. All tosses are assumed independent.

(a) Find the PMF, the expected value, and the variance of the number of tosses.

(b) What is the probability that the last toss of the first coin is a head?

Problem 2. **Laplace random variable.** Let X have the PDF

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|},$$

where λ is a positive scalar. Verify that f_X satisfies the normalization condition, and evaluate the mean and variance of X.