# ELE 273 HOMEWORK 5 SOLUTIONS

**Solution to Problem 4.2.** Let $Y = e^X$. We first find the CDF of $Y$, and then take the derivative to find its PDF. We have

$$\mathbf{P}(Y \le y) = \mathbf{P}(e^X \le y) = \begin{cases} \mathbf{P}(X \le \ln y), & \text{if } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} f_Y(y) &= \begin{cases} \dfrac{d}{dx} F_X(\ln y), & \text{if } y > 0, \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} \dfrac{1}{y} f_X(\ln y), & \text{if } y > 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

When $X$ is uniform on $[0, 1]$, the answer simplifies to

$$f_Y(y) = \begin{cases} \dfrac{1}{y}, & \text{if } 0 < y \le e, \\ 0, & \text{otherwise.} \end{cases}$$

**Solution to Problem 4.10.** We first note that the range of possible values of $Z$ are the integers from the range $[1, 5]$. Thus we have

$$p_Z(z) = 0, \quad \text{if } z \ne 1, 2, 3, 4, 5.$$

We calculate $p_Z(z)$ for each of the values $z = 1, 2, 3, 4, 5$, using the convolution formula. We have

$$p_Z(1) = \sum_x p_X(x) p_Y(1 - x) = p_X(1) p_Y(0) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6},$$

where the second equality above is based on the fact that for $x \ne 1$ either $p_X(x)$ or $p_Y(1 - x)$ (or both) is zero. Similarly, we obtain

$$p_Z(2) = p_X(1) p_Y(1) + p_X(2) p_Y(0) = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{18},$$

$$p_Z(3) = p_X(1) p_Y(2) + p_X(2) p_Y(1) + p_X(3) p_Y(0) = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3},$$

$$p_Z(4) = p_X(2) p_Y(2) + p_X(3) p_Y(1) = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{6},$$

$$p_Z(5) = p_X(3) p_Y(2) = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}.$$

**Solution to Problem 4.12.** Let $V = X + Y$. As in Example 4.10, the PDF of $V$ is

$$f_V(v) = \begin{cases} v, & 0 \le v \le 1, \\ 2 - v, & 1 \le v \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let $W = X + Y + Z = V + Z$. We convolve the PDFs $f_V$ and $f_Z$, to obtain

$$f_W(w) = \int f_V(v) f_Z(w - v)\, dv.$$

We first need to determine the limits of the integration. Since $f_V(v) = 0$ outside the range $0 \le v \le 2$, and $f_W(w - v) = 0$ outside the range $0 \le w - v \le 1$, we see that the integrand can be nonzero only if

$$0 \le v \le 2, \qquad \text{and} \qquad w - 1 \le v \le w.$$

We consider three separate cases. If $w \le 1$, we have

$$f_W(w) = \int_0^w f_V(v) f_Z(w - v)\, dv = \int_0^w v\, dv = \frac{w^2}{2}.$$

If $1 \le w \le 2$, we have

$$\begin{aligned} f_W(w) &= \int_{w-1}^{w} f_V(v) f_Z(w - v)\, dv \\ &= \int_{w-1}^{1} v\, dv + \int_1^w (2 - v)\, dv \\ &= \frac{1}{2} - \frac{(w-1)^2}{2} - \frac{(w-2)^2}{2} + \frac{1}{2}. \end{aligned}$$

Finally, if $2 \le w \le 3$, we have

$$f_W(w) = \int_{w-1}^{2} f_V(v) f_Z(w - v)\, dv = \int_{w-1}^{2} (2 - v)\, dv = \frac{(3-w)^2}{2}.$$

To summarize,

$$f_W(w) = \begin{cases} w^2/2, & 0 \le w \le 1, \\ 1 - (w-1)^2/2 - (2-w)^2/2, & 1 \le w \le 2, \\ (3-w)^2/2, & 2 \le w \le 3, \\ 0, & \text{otherwise}. \end{cases}$$

**Solution to Problem 4.17.** Because the covariance remains unchanged when we add a constant to a random variable, we can assume without loss of generality that $X$ and $Y$ have zero mean. We then have

$$\text{cov}(X - Y, X + Y) = \mathbf{E}\big[(X - Y)(X + Y)\big] = \mathbf{E}[X^2] - \mathbf{E}[Y^2] = \text{var}(X) - \text{var}(Y) = 0,$$

since $X$ and $Y$ were assumed to have the same variance.

**Solution to Problem 4.18.** We have

$$\text{cov}(R, S) = \mathbf{E}[RS] - \mathbf{E}[R]\mathbf{E}[S] = \mathbf{E}[WX + WY + X^2 + XY] = \mathbf{E}[X^2] = 1,$$

and

$$\text{var}(R) = \text{var}(S) = 2,$$

so

$$\rho(R, S) = \frac{\text{cov}(R, S)}{\sqrt{\text{var}(R)\text{var}(S)}} = \frac{1}{2}.$$

We also have

$$\text{cov}(R, T) = \mathbf{E}[RT] - \mathbf{E}[R]\mathbf{E}[T] = \mathbf{E}[WY + WZ + XY + XZ] = 0,$$

so that

$$\rho(R, T) = 0.$$

**Solution to Problem 4.19.** To compute the correlation coefficient

$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y},$$

we first compute the covariance:

$$\begin{aligned}
\text{cov}(X,Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= \mathbf{E}[aX + bX^2 + cX^3] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= a\mathbf{E}[X] + b\mathbf{E}[X^2] + c\mathbf{E}[X^3] \\
&= b.
\end{aligned}$$

We also have

$$\begin{aligned}
\text{var}(Y) &= \text{var}(a + bX + cX^2) \\
&= \mathbf{E}\big[(a + bX + cX^2)^2\big] - \big(\mathbf{E}[a + bX + cX^2]\big)^2 \\
&= (a^2 + 2ac + b^2 + 3c^2) - (a^2 + c^2 + 2ac) \\
&= b^2 + 2c^2,
\end{aligned}$$

and therefore, using the fact $\text{var}(X) = 1$,

$$\rho(X,Y) = \frac{b}{\sqrt{b^2 + 2c^2}}.$$

**Solution to Problem 4.24.** (a) Consider the following two random variables:

$X$ = amount of time the professor devotes to his task [exponentially distributed with parameter $\lambda(y) = 1/(5-y)$];

$Y$ = length of time between 9 a.m. and his arrival (uniformly distributed between 0 and 4).

Note that $\mathbf{E}[Y] = 2$. We have

$$\mathbf{E}[X \mid Y = y] = \frac{1}{\lambda(y)} = 5 - y,$$

which implies that

$$\mathbf{E}[X \mid Y] = 5 - Y,$$

and

$$\mathbf{E}[X] = \mathbf{E}\big[\mathbf{E}[X \mid Y]\big] = \mathbf{E}[5 - Y] = 5 - \mathbf{E}[Y] = 5 - 2 = 3.$$

(b) Let $Z$ be the length of time from 9 a.m. until the professor completes the task. Then,

$$Z = X + Y.$$

We already know from part (a) that $\mathbf{E}[X] = 3$ and $\mathbf{E}[Y] = 2$, so that

$$\mathbf{E}[Z] = \mathbf{E}[X] + \mathbf{E}[Y] = 3 + 2 = 5.$$

Thus the expected time that the professor leaves his office is 5 hours after 9 a.m.

(c) We define the following random variables:

$W$ = length of time between 9 a.m. and arrival of the Ph.D. student (uniformly distributed between 9 a.m. and 5 p.m.).

$R$ = amount of time the student will spend with the professor, if he finds the professor (uniformly distributed between 0 and 1 hour).

$T$ = amount of time the professor will spend with the student.

Let also $F$ be the event that the student finds the professor.

To find $\mathbf{E}[T]$, we write

$$\mathbf{E}[T] = \mathbf{P}(F)\mathbf{E}[T \mid F] + \mathbf{P}(F^c)\mathbf{E}[T \mid F^c]$$

Using the problem data,

$$\mathbf{E}[T \mid F] = \mathbf{E}[R] = \frac{1}{2}$$

(this is the expected value of a uniformly distribution ranging from 0 to 1),

$$\mathbf{E}[T \mid F^c] = 0$$

(since the student leaves if he does not find the professor). We have

$$\mathbf{E}[T] = \mathbf{E}[T \mid F]\mathbf{P}(F) = \frac{1}{2}\mathbf{P}(F),$$

so we need to find $\mathbf{P}(F)$.

In order for the student to find the professor, his arrival should be between the arrival and the departure of the professor. Thus

$$\mathbf{P}(F) = \mathbf{P}(Y \le W \le X + Y).$$

We have that $W$ can be between 0 (9 a.m.) and 8 (5 p.m.), but $X+Y$ can be any value greater than 0. In particular, it may happen that the sum is greater than the upper bound for $W$. We write

$$\mathbf{P}(F) = \mathbf{P}(Y \le W \le X+Y) = 1 - \big(\mathbf{P}(W < Y) + \mathbf{P}(W > X+Y)\big)$$

We have

$$\mathbf{P}(W < Y) = \int_0^4 \frac{1}{4} \int_0^y \frac{1}{8}\, dw\, dy = \frac{1}{4}$$

and

$$\begin{aligned}
\mathbf{P}(W > X+Y) &= \int_0^4 \mathbf{P}(W > X+Y \,|\, Y=y) f_Y(y)\, dy \\
&= \int_0^4 \mathbf{P}(X < W-Y \,|\, Y=y) f_Y(y)\, dy \\
&= \int_0^4 \int_y^8 F_{X|Y}(w-y) f_W(w) f_Y(y)\, dw\, dy \\
&= \int_0^4 \frac{1}{4} \int_y^8 \frac{1}{8} \int_0^{w-y} \frac{1}{5-y} e^{-\frac{x}{5-y}}\, dx\, dw\, dy \\
&= \frac{12}{32} + \frac{1}{32} \int_0^4 (5-y) e^{-\frac{8-y}{5-y}}\, dy.
\end{aligned}$$

Integrating numerically, we have

$$\int_0^4 (5-y) e^{-\frac{8-y}{5-y}}\, dy = 1.7584.$$

Thus,

$$\mathbf{P}(Y \le W \le X+Y) = 1 - \big(\mathbf{P}(W < Y) + \mathbf{P}(W > X+Y)\big) = 1 - 0.68 = 0.32.$$

The expected amount of time the professor will spend with the student is then

$$\mathbf{E}[T] = \frac{1}{2}\mathbf{P}(F) = \frac{1}{2}\, 0.32 = 0.16 = 9.6 \text{ mins.}$$

Next, we want to find the expected time the professor will leave his office. Let $Z$ be the length of time measured from 9 a.m. until he leaves his office. If the professor doesn't spend any time with the student, then $Z$ will be equal to $X+Y$. On the other hand, if the professor is interrupted by the student, then the length of time will be equal to $X+Y+R$. This is because the professor will spend the same amount of total time on the task regardless of whether he is interrupted by the student. Therefore,

$$\mathbf{E}[Z] = \mathbf{P}(F)\mathbf{E}[Z \,|\, F] + \mathbf{P}(F^c)\mathbf{E}[Z \,|\, F^c] = \mathbf{P}(F)\mathbf{E}[X+Y+R] + \mathbf{P}(F^c)\mathbf{E}[X+Y].$$

Using the results of the earlier calculations,

$$\mathbf{E}[X+Y] = 5,$$

$$\mathbf{E}[X+Y+R] = \mathbf{E}[X+Y] + \mathbf{E}[R] = 5 + \frac{1}{2} = \frac{11}{2}.$$

Therefore,

$$\mathbf{E}[Z] = 0.68 \cdot 5 + 0.32 \cdot \frac{11}{2} = 5.16.$$

Thus the expected time the professor will leave his office is 5.16 hours after 9 a.m.