ELE 273 HOMEWORK 6 SOLUTIONS

Solution to Problem 4.30. The transform associated with X is

$$M_X(s) = e^{s^2/2}.$$

By taking derivatives with respect to s, we find that

$$\mathbf{E}[X] = 0, \quad \mathbf{E}[X^2] = 1, \quad \mathbf{E}[X^3] = 0, \quad \mathbf{E}[X^4] = 3.$$

Solution to Problem 4.33. We recognize this transform as corresponding to the following mixture of exponential PDFs:

$$f_X(x) = \begin{cases} \frac{1}{3} \cdot 2e^{-2x} + \frac{2}{3} \cdot 3e^{-3x}, & \text{for } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

By the inversion theorem, this must be the desired PDF.

Solution to Problem 4.36. (a) We have $U = X$ if $X = 1$, which happens with probability $1/3$, and $U = Z$ if $X = 0$, which happens with probability $2/3$. Therefore, U is a mixture of random variables and the associated transform is

$$M_U(s) = \mathbf{P}(X = 1)M_Y(s) + \mathbf{P}(X = 0)M_Z(s) = \frac{1}{3} \cdot \frac{2}{2-s} + \frac{2}{3}e^{3(e^s-1)}.$$

(b) Let $V = 2Z + 3$. We have

$$M_V(s) = e^{3s}M_Z(2s) = e^{3s}e^{3(e^{2s}-1)} = e^{3(s-1+e^{2s})}.$$

(c) Let $W = Y + Z$. We have

$$M_W(s) = M_Y(s)M_Z(s) = \frac{2}{2-s}e^{3(e^s-1)}.$$

Solution to Problem 4.43. (a) Using the total probability theorem, we have

$$\mathbf{P}(X > 4) = \sum_{k=0}^{4} \mathbf{P}(k \text{ lights are red})\mathbf{P}(X > 4 \,|\, k \text{ lights are red}).$$

We have

$$\mathbf{P}(k \text{ lights are red}) = \binom{4}{k}\left(\frac{1}{2}\right)^4.$$

The conditional PDF of X given that k lights are red, is normal with mean k minutes and standard deviation $(1/2)\sqrt{k}$. Thus, X is a mixture of normal random variables and the transform associated with its (unconditional) PDF is the corresponding mixture of the transforms associated with the (conditional) normal PDFs. However, X is not normal, because a mixture of normal PDFs need not be normal. The probability $\mathbf{P}(X > 4 \,|\, k \text{ lights are red})$ can be computed from the normal tables for each k, and $\mathbf{P}(X > 4)$ is obtained by substituting the results in the total probability formula above.

(b) Let K be the number of traffic lights that are found to be red. We can view X as the sum of K independent normal random variables. Thus the transform associated with X can be found by replacing in the binomial transform $M_K(s) = (1/2 + (1/2)e^s)^4$ the occurrence of e^s by the normal transform corresponding to $\mu = 1$ and $\sigma = 1/2$. Thus

$$M_X(s) = \left(\frac{1}{2} + \frac{1}{2}\left(e^{\frac{(1/2)^2 s^2}{2} + s}\right)\right)^4.$$

Note that by using the formula for the transform, we cannot easily obtain the probability $\mathbf{P}(X > 4)$.

Solution to Problem 5.1. (a) We have $\sigma_{M_n} = 1/\sqrt{n}$, so in order that $\sigma_{M_n} \leq 0.01$, we must have $n \geq 10{,}000$.

(b) We want to have

$$\mathbf{P}\big(|M_n - h| \leq 0.05\big) \geq 0.99.$$

Using the facts $h = \mathbf{E}[M_n]$, $\sigma^2_{M_n} = 1/n$, and the Chebyshev inequality, we have

$$\begin{aligned}\mathbf{P}\big(|M_n - h| \leq 0.05\big) &= \mathbf{P}\big(|M_n - \mathbf{E}[M_n]| \leq 0.05\big)\\ &= 1 - \mathbf{P}\big(|M_n - \mathbf{E}[M_n]| \geq 0.05\big)\\ &\geq 1 - \frac{1/n}{(0.05)^2}.\end{aligned}$$

Thus, we must have

$$1 - \frac{1/n}{(0.05)^2} \geq 0.99,$$

which yields $n \geq 40{,}000$.

(c) Based on Example 5.3, $\sigma^2_{X_i} \leq (0.6)^2/4$, so he should use 0.3 meters in place of 1.0 meters as the estimate of the standard deviation of the samples X_i in the calculations of parts (a) and (b). In the case of part (a), we have $\sigma_{M_n} = 0.3/\sqrt{n}$, so in order that $\sigma_{M_n} \leq 0.01$, we must have $n \geq 900$. In the case of part (b), we have $\sigma_{M_n} = 0.3/\sqrt{n}$, so in order that $\sigma_{M_n} \leq 0.01$, we must have $n \geq 900$. In the case of part (a), we must have

$$1 - \frac{0.09/n}{(0.05)^2} \geq 0.99,$$

which yields $n \geq 3{,}600$.

Solution to Problem 5.9. (a) Let S be the number of crash-free days, which is a binomial random variable with parameters $n = 50$ and $p = 0.95$, so that $\mathbf{E}[X] = 50 \cdot 0.95 = 47.5$ and $\sigma_S = \sqrt{50 \cdot 0.95 \cdot 0.05} = 1.54$. Using the normal approximation to the binomial, we find

$$\mathbf{P}(S \geq 45) = \mathbf{P}\left(\frac{S - 47.5}{1.54} \geq \frac{45 - 47.5}{1.54}\right) \approx 1 - \Phi(-1.62) = \Phi(1.62) = 0.9474.$$

A better approximation can be obtained by using the de Moivre-Laplace approximation, which yields

$$\mathbf{P}(S \geq 45) = \mathbf{P}(S > 44.5) = \mathbf{P}\left(\frac{S - 47.5}{1.54} \geq \frac{44.5 - 47.5}{1.54}\right)$$
$$\approx 1 - \Phi(-1.95) = \Phi(1.95) = 0.9744.$$

(b) The random variable S is binomial with parameter $p = 0.95$. However, the random variable $50 - S$ (the number of crashes) is also binomial with parameter $p = 0.05$. Since the Poisson approximation is exact in the limit of small p and large n, it will give more accurate results if applied to $50 - S$. We will therefore approximate $50 - S$ by a Poisson random variable with parameter $\lambda = 50 \cdot 0.05 = 2.5$. Thus,

$$\begin{aligned}\mathbf{P}(S \geq 45) &= \mathbf{P}(50 - S \leq 5) \\ &= \sum_{k=0}^{5} \mathbf{P}(n - S = k) \\ &= \sum_{k=0}^{5} e^{-\lambda} \frac{\lambda^k}{k!} \\ &= 0.958.\end{aligned}$$

It is instructive to compare with the exact probability which is

$$\sum_{k=0}^{5} \binom{50}{k} 0.05^k \cdot 0.95^{50-k} = 0.962.$$

Thus, the Poisson approximation is closer. This is consistent with the intuition that the normal approximation to the binomial works well when p is close to 0.5 or n is very large, which is not the case here. On the other hand, the calculations based on the normal approximation are generally less tedious.