Assignment -2 (Solutions)

S1)

S1.1) Gauss Elimination Method

$$\begin{bmatrix} 1 & 4 & 2 & 9 \\ 5 & 1 & 1 & 7 \\ 1 & 2 & 3 & 5 \end{bmatrix} \xrightarrow{-5s1+s2=s2} \begin{bmatrix} 1 & 4 & 2 & 9 \\ 0 & -19 & -9 & -38 \\ 1 & 2 & 3 & 5 \end{bmatrix} \xrightarrow{-s1+s3=s3}$$

$$\begin{bmatrix} 1 & 4 & 2 & 9 \\ 0 & -19 & -9 & -38 \\ 0 & -2 & 1 & 4 \end{bmatrix} \xrightarrow{-2/19s2+s3=s3} \begin{bmatrix} 1 & 4 & 2 & 9 \\ 0 & -19 & -9 & -38 \\ 0 & 0 & 37/19 & 0 \end{bmatrix}$$

$$\frac{37}{19}x_3 = 0 \Rightarrow x_3 = 0, -19x_2 - 9x_3 = -38 \Rightarrow x_2 = 2, x_1 + 4x_2 + 2x_3 = 9 \Rightarrow x_1 = 1$$

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

S1.2) LU Method

$$\begin{bmatrix}1 & 4 & 2\\5 & 1 & 1\\1 & 2 & 3\end{bmatrix} \xrightarrow{s2-5s1=s2} \begin{bmatrix}1 & 4 & 2\\0 & -19 & -9\\1 & 2 & 3\end{bmatrix} \xrightarrow{s3-s1=s3}$$

$$\begin{bmatrix}1 & 4 & 2\\0 & -19 & -9\\0 & -2 & 1\end{bmatrix} \xrightarrow{s3+2/19s2=s3} \underbrace{\begin{bmatrix}1 & 4 & 2\\0 & -19 & -9\\0 & 0 & \frac{37}{19}\end{bmatrix}}_{u} \underbrace{\begin{bmatrix}1 & 0 & 0\\5 & 1 & 0\\1 & \frac{2}{19} & 1\end{bmatrix}}_{L} \underbrace{\begin{bmatrix}9\\7\\5\end{bmatrix}}_{b}$$

$Lu\vec{x}=\vec{b}$ $\quad L\vec{y}=\vec{b}$ $\quad U\vec{x}=\vec{y}$

$$\begin{bmatrix}1 & 0 & 0\\5 & -1 & 0\\1 & 2/19 & 1\end{bmatrix} \cdot \vec{y} = \underbrace{\begin{bmatrix}9\\7\\5\end{bmatrix}}_{b}$$

$$\begin{bmatrix}1 & 4 & 2\\0 & -19 & -9\\0 & 0 & 37/19\end{bmatrix} \cdot \vec{x} = \begin{bmatrix}9\\-38\\0\end{bmatrix} \Rightarrow x = \begin{bmatrix}1\\2\\0\end{bmatrix}$$

S1.3) Choleski Method

$$\begin{bmatrix} 1 & 4 & 2 \\ 5 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{s2 \rightleftarrows s1} \underbrace{\begin{bmatrix} 5 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}}_{A} \quad \text{(Pos \& Symmetric matrix)}$$

$$L.L^T = A \qquad L.\underbrace{L^T.\vec{x}}_{\vec{y}} = \vec{b}$$

$$L_{11} = \sqrt{A_{11}}, L_{21} = \frac{\sqrt{A_{21}}}{L_{11}}, L_{31} = \frac{A_{31}}{L_{11}}$$

$$L_{22} = \sqrt{A_{22} - (\mathrm{L}_{21})^2}, L_{33} = \sqrt{A_{33} - (\mathrm{L}_{31}{}^2 + \mathrm{L}_{32}{}^2)^2}$$

$$L_{32} = \frac{A_{33} - \mathrm{L}_{31}.\mathrm{L}_{21}}{\mathrm{L}_{22}}$$

$$A.\vec{x} = \vec{b}$$

$$L.\vec{y} = \vec{b} \Rightarrow \begin{bmatrix} 2.23 & 0 & 0 \\ 0.44 & 2 & 0 \\ 0.44 & 0.9 & 1.42 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 5 \end{bmatrix}$$

$$L^T.\vec{x} = \vec{y} \Rightarrow \begin{bmatrix} 2.2 & 0.45 & 0.45 \\ 0.44 & 2 & 0.9 \\ 0.44 & 0 & 1.41 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3.13 \\ 3.8 \\ 0.13 \end{bmatrix}$$

$$\cancel{\vec{x} = \begin{bmatrix} 1 \\ 1.86 \\ 0.093 \end{bmatrix} \cong \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}} \qquad X = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

S1.4) Gauss Seidel

$$\underbrace{\begin{bmatrix} 5 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 7 \\ 7 \\ 5 \end{bmatrix}}_{\vec{b}} \Rightarrow$$

$$x_1 = \frac{7 - x_2 - x_3}{5}$$
$$x_2 = \frac{9 - x_2 - 2x_3}{4}$$
$$x_3 = \frac{5 - 2x_1 - 2x_2}{3}$$

$$x_1^{(1)}(x_2^{(1)} = x_3^{(1)} = 0) = \frac{9}{4}, x_2^{(1)}(x_1^{(1)} = \frac{9}{4}, x_3^{(1)} = 0) = 0.95, x_3^{(1)}(x_1^{(1)} = \frac{9}{4}, x_2^{(1)} = 0.95) = -0.15$$

$$x_1^{(2)}(x_2^{(1)} = 0.95, x_3^{(1)} = -0.15) = 2.09, x_2^{(2)}(x_1^{(2)} = 2.09, x_3^{(2)} = -0.15) = 1.012, x_3^{(2)}(x_1^{(2)} = 2.09, x_2^{(2)} = 1.012) = -0.064$$

$$\vec{\mathrm{x}} = \begin{bmatrix} 2.09 \\ 1.012 \\ -0.064 \end{bmatrix}$$

1.5) CGM

$$A=\begin{bmatrix}5&1&1\\1&4&2\\1&2&3\end{bmatrix} \quad \vec{b}=\begin{bmatrix}7\\9\\5\end{bmatrix}$$

İlk iterasyon için:

$$x_0=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$r_0=b-Ax_0=\begin{bmatrix}7\\9\\5\end{bmatrix}-\begin{bmatrix}5&1&1\\1&4&2\\1&2&3\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}7\\9\\5\end{bmatrix}$$

$$s_0=r_0=\begin{bmatrix}7\\9\\5\end{bmatrix}$$

$$\alpha_0=\frac{s_0^T(b-Ax_0)}{s_0^T A s_0}=\frac{s_0^T\cdot r_0}{s_0^T A s_0}=\frac{[7\ 9\ 5]\begin{bmatrix}7\\9\\5\end{bmatrix}}{[7\ 9\ 5]\begin{bmatrix}5&1&1\\1&4&2\\1&2&3\end{bmatrix}\begin{bmatrix}7\\9\\5\end{bmatrix}}=\frac{[49+81+25]}{[49\ 53\ 40]\begin{bmatrix}7\\9\\5\end{bmatrix}}=\frac{[155]}{[1020]}=0{,}152$$

$$\vec{x}_1=\vec{x}_0+\alpha_0\vec{s}_0=\begin{bmatrix}0\\0\\0\end{bmatrix}+0{,}152\begin{bmatrix}7\\9\\5\end{bmatrix}=\begin{bmatrix}1{,}064\\1{,}368\\0{,}76\end{bmatrix}$$

İkinci iterasyon için:

$$\vec{x}_2=\vec{x}_1+\alpha\vec{s}_1$$

$$\alpha_1=\frac{s_1^T(b-A\vec{x}_1)}{s_1^T A s_1}$$

$$\vec{s}_1=\vec{r}_1+\vec{\beta}_0\vec{s}_0$$

$$\vec{\beta}_0=\frac{-r_1^T A s_0}{s_0^T A s_0}=\frac{(b-Ax_1)^T A s_0}{s_0^T A s_0}=\frac{\left(\begin{bmatrix}7\\9\\5\end{bmatrix}-\begin{bmatrix}5&1&1\\1&4&2\\1&2&3\end{bmatrix}\begin{bmatrix}1{,}064\\1{,}368\\0{,}76\end{bmatrix}\right)^T\begin{bmatrix}5&1&1\\1&4&2\\1&2&3\end{bmatrix}\begin{bmatrix}7\\9\\5\end{bmatrix}}{[7\ 9\ 5]\begin{bmatrix}5&1&1\\1&4&2\\1&2&3\end{bmatrix}\begin{bmatrix}7\\9\\5\end{bmatrix}}=\frac{\begin{bmatrix}-0{,}448\\0{,}944\\-1{,}08\end{bmatrix}^T\begin{bmatrix}5&1&1\\1&4&2\\1&2&3\end{bmatrix}\begin{bmatrix}7\\9\\5\end{bmatrix}}{[7\ 9\ 5]\begin{bmatrix}5&1&1\\1&4&2\\1&2&3\end{bmatrix}\begin{bmatrix}7\\9\\5\end{bmatrix}}$$

$$=\frac{[-0{,}448\ \ 0{,}944\ \ -1{,}08]\begin{bmatrix}5&1&1\\1&4&2\\1&2&3\end{bmatrix}\begin{bmatrix}7\\9\\5\end{bmatrix}}{[7\ 9\ 5]\begin{bmatrix}5&1&1\\1&4&2\\1&2&3\end{bmatrix}\begin{bmatrix}7\\9\\5\end{bmatrix}}=\frac{[-15{,}12]}{[1020]}=-0{,}0149$$

$$\vec{s}_1=\begin{bmatrix}-0{,}448\\0{,}944\\-1{,}08\end{bmatrix}-0{,}0149\begin{bmatrix}7\\9\\5\end{bmatrix}$$

$$\vec{s}_1=\begin{bmatrix}-0{,}55\\0{,}81\\-1{,}155\end{bmatrix} \quad \vec{\alpha}_1=\frac{[-0{,}55\ \ 0{,}81\ \ -1{,}155]\begin{bmatrix}-0{,}448\\0{,}944\\-1{,}08\end{bmatrix}}{[-0{,}55\ \ 0{,}81\ \ -1{,}155]\begin{bmatrix}5&1&1\\1&4&2\\1&2&3\end{bmatrix}\begin{bmatrix}-0{,}55\\0{,}81\\-1{,}155\end{bmatrix}}=\frac{[2{,}26]}{[4{,}782]}=0{,}473$$

$$\vec{x}_2=\begin{bmatrix}1{,}064\\1{,}368\\0{,}76\end{bmatrix}+0{,}473\begin{bmatrix}-0{,}55\\0{,}81\\-1{,}155\end{bmatrix}=\begin{bmatrix}1{,}064\\1{,}368\\0{,}76\end{bmatrix}+\begin{bmatrix}-0{,}26\\0{,}383\\-0{,}546\end{bmatrix}=\begin{bmatrix}0{,}804\\1{,}751\\0{,}214\end{bmatrix}\cong\begin{bmatrix}1\\2\\0\end{bmatrix}$$

2) (a_x, a_y) (b_x, b_y), (c_x, c_y)

Langrange yöntemi:

$$l_1(x) = \frac{(x-b_x)(x-c_x)}{(a_x-b_x)(a_x-c_x)}$$

$$l_2(x) = \frac{(x-a_x)(x-c_x)}{(b_x-a_x)(b_x-c_x)}$$

$$l_3(x) = \frac{(x-a_x)(x-b_x)}{(c_x-a_x)(c_x-b_x)}$$

$$p(x) = a_y l_1(x) + b_y l_2(x) + c_y l_3(x)$$

$$\boxed{p(x) = a_y \frac{(x-b_x)(x-c_x)}{(a_x-b_x)(a_x-c_x)} + b_y \frac{(x-a_x)(x-c_x)}{(b_x-a_x)(b_x-c_x)} + c_y \frac{(x-a_x)(x-b_x)}{(c_x-a_x)(c_x-b_x)}}$$

Neville yöntemi:

$$p_0[x_1] = a_y \quad p_0[x_2] = b_y \quad p_0[x_3] = c_y$$

$$p_1[x_1, x_2] = \frac{(x-b_x)a_y - b_y(x-a_x)}{a_x-b_x}$$

$$p_1[x_2, x_3] = \frac{(x-c_x)b_y - c_y(x-b_x)}{b_x-c_x}$$

$$p_1[x_1, x_2, x_3] = \frac{(x-c_x)\dfrac{a_y(x-b_x) - b_y(x-a_x)}{a_x-b_x} - (x-a_x)\dfrac{b_y(x-c_x) - c_y(x-b_x)}{b_x-c_x}}{a_x-c_x}$$

$$= a_y \frac{(x-b_x)(x-c_x)}{(a_x-b_x)(a_x-c_x)} \underbrace{- b_y \left[\frac{(x-a_x)(x-c_x)}{(a_x-b_x)(a_x-c_x)} + \frac{(x-c_x)(x-a_x)}{(b_x-c_x)(a_x-c_x)} \right]} + c_y \frac{(x-b_x)(x-a_x)}{\cancel{-}(c_x-a_x) \cdot \cancel{-}(c_x-b_x)}$$

$$= \qquad -b_y \frac{(x-a_x)(x-c_x)}{(a_x-c_x)} \underbrace{\left(\frac{1}{a_x-b_x} + \frac{1}{b_x-c_x} \right)}_{\dfrac{a_x-c_x}{(a_x-b_x)(b_x-c_x)}} \quad -b_y \frac{(x-a_x)(x-c_x)}{(a_x-\cancel{c_x})} \cdot \frac{(a_x-\cancel{c_x})}{(a_x-b_x)(b_x-c_x)} \quad - b_y \frac{(x-a_x)(x-c_x)}{-(b_x-a_x)(b_x-c_x)}$$

$$\boxed{= a_y \frac{(x-b_x)(x-c_x)}{(a_x-b_x)(a_x-c_x)} + b_y \frac{(x-a_x)(x-c_x)}{(b_x-a_x)(b_x-c_x)} + c_y \cdot \frac{(x-b_x)(x-a_x)}{(c_x-a_x)(c_x-b_x)}}$$

Newton yöntemi:

$$P(x) = P(a_x) + (x-a_x)P(a_x, b_x) + (x-a_x)(x-b_x)P(a_x, b_x, c_x)$$

$$P(a_x,b_x) = \frac{b_y - a_y}{b_x - a_x} \qquad P(b_x,c_x) = \frac{c_y - b_y}{c_x - b_x} \qquad P(a_x,b_x,c_x) = \frac{P(b_x,c_x) - P(a_x,b_x)}{c_x - a_x}$$

$$P(x) = a_y + \frac{(x-a_x)(b_y-a_y)}{(b_x-a_x)} + \underbrace{\frac{(x-a_x)(x-b_x)}{(c_x-a_x)}\left[\frac{(c_y-b_y)}{(c_x-b_x)} - \frac{(b_y-a_y)}{(b_x-a_x)}\right]}$$

$$\frac{(x-a_x)(x-b_x)}{(c_x-a_x)}\left[\frac{c_y b_x - c_y a_x - \cancel{b_y b_x} + b_y a_x - b_y c_x + \cancel{b_y b_x} + a_y c_x - a_y b_x}{(c_x-b_x)(b_x-a_x)}\right]$$

$$\frac{(x-a_x)(x-b_x)}{(c_x-a_x)}\left[\frac{c_y(b_x-a_x) + b_y(a_x-c_x) + a_y(c_x-b_x)}{(c_x-b_x)(b_x-a_x)}\right]$$

$$\frac{a_y\cancel{(c_x-b_x)}(x-a_x)(x-b_x)}{\cancel{(c_x-b_x)}(c_x-a_x)(b_x-a_x)} + \frac{b_y\overset{-1}{\cancel{(a_x-c_x)}}(x-a_x)(x-b_x)}{\cancel{(c_x-a_x)}(c_x-b_x)(b_x-a_x)} + \frac{c_y\cancel{(b_x-a_x)}(x-a_x)(x-b_x)}{(c_x-a_x)(c_x-b_x)\cancel{(b_x-a_x)}}$$

$$a_y\frac{(x-a_x)(x-b_x)}{(c_x-a_x)(b_x-a_x)} - b_y\frac{(x-a_x)(x-b_x)}{(c_x-b_x)(b_x-a_x)} + c_y\frac{(x-a_x)(x-b_x)}{(c_x-a_x)(c_x-b_x)}$$

$$P(x) = \underbrace{a_y} \underbrace{- a_y\frac{(x-a_x)}{(b_x-a_x)}} \underbrace{+ b_y\frac{(x-a_x)}{(b_x-a_x)}} \underbrace{+ a_y\frac{(x-a_x)(x-b_x)}{(c_x-a_x)(b_x-a_x)}} \underbrace{- b_y\frac{(x-a_x)(x-b_x)}{(c_x-b_x)(b_x-a_x)}} \underbrace{+ c_y\frac{(x-a_x)(x-b_x)}{(c_x-a_x)(c_x-b_x)}}$$

$*$ $$a_y\left(1 - \frac{(x-a_x)}{(b_x-a_x)} + \frac{(x-a_x)(x-b_x)}{(c_x-a_x)(b_x-a_x)}\right) = a_y\left[\frac{(c_x-a_x)(b_x-a_x) - (x-a_x)(c_x-a_x) + (x-a_x)(x-b_x)}{(c_x-a_x)(b_x-a_x)}\right]$$

$$a_y\left[\frac{c_x b_x - \cancel{c_x a_x} - \cancel{b_x a_x} + \cancel{a_x^2} - x c_x + \cancel{x a_x} + \cancel{a_x c_x} - \cancel{a_x^2} + x^2 - x b_x - \cancel{x a_x} + \cancel{a_x b_x}}{(c_x-a_x)(b_x-a_x)}\right]$$

$$a_y\frac{(c_x b_x - x c_x + x^2 - x b_x)}{(c_x-a_x)(b_x-a_x)} = a_y\left[\frac{c_x(b_x-x) + x(x-b_x)}{(c_x-a_x)(b_x-a_x)}\right] = a_y\frac{(x-b_x)(x-c_x)}{(a_x-c_x)(a_x-b_x)}$$

$*$ $$b_y\left[\frac{(x-a_x)}{(b_x-a_x)} - \frac{(x-a_x)(x-b_x)}{(c_x-b_x)(b_x-a_x)}\right] = b_y\left[\frac{(x-a_x)(c_x-b_x) - (x-a_x)(x-b_x)}{(c_x-b_x)(b_x-a_x)}\right]$$

$$b_y\left[\frac{x c_x - x b_x - a_x c_x + a_x b_x - x^2 + x b_x + x a_x - a_x b_x}{(x-b_x)(b_x-a_x)}\right] = b_y\frac{(x-a_x)(x-c_x)}{(b_x-c_x)(b_x-a_x)}$$

$*$ $$c_y\left[\frac{(x-a_x)(x-b_x)}{(c_x-a_x)(c_x-b_x)}\right]$$

$$\boxed{P(x) = a_y\frac{(x-b_x)(x-c_x)}{(a_x-c_x)(a_x-b_x)} + b_y\frac{(x-a_x)(x-c_x)}{(b_x-c_x)(b_x-a_x)} + c_y\frac{(x-a_x)(x-b_x)}{(c_x-a_x)(c_x-b_x)}}$$

3) kübik Spline

	①	②	③	④	⑤
x:	1	2	3	4	5
y:	0	1	0	1	0

$f(1,5)$

$k_{i-1} + 4k_i + k_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$ $i = 2, \dots, n-1$ $h=1$ $k_1 = k_n = 0$

$i=2 \Rightarrow$ $k_1 + 4k_2 + k_3 = 6(\overset{0}{\cancel{y_1}} - 2y_2 + \overset{0}{\cancel{y_3}})$ $\boxed{k_1 = k_5 = 0}$

$\underbrace{\overset{0}{\cancel{k_1}} + 4k_2 + k_3 = -12}$ $\rightarrow$ $4k_2 + k_3 = -12$

$i=3$ $k_2 + 4k_3 + k_4 = 6(y_2 - 2\overset{0}{\cancel{y_3}} + y_4)$

$\underbrace{k_2 + 4k_3 + k_4 = 12}$ $\rightarrow$ $k_2 + 4k_3 + k_4 = 12$

$i=4$ $k_3 + 4k_4 + k_5 = 6(\overset{0}{\cancel{y_3}} - 2y_4 + \overset{0}{\cancel{y_5}})$

$\underbrace{k_3 + 4k_4 + \overset{0}{\cancel{k_5}} = -12}$ $\rightarrow$ $k_3 + 4k_4 = -12$

$4k_2 + k_3 = -12$

$-4/\ k_2 + 4k_3 + k_4 = 12$

$+$

$k_3 - 16k_3 - 4k_4 = -60$

$\underbrace{-15k_3 - 4k_4 = -60}$

$-15k_3 - 4k_4 = -60$

$+\ \ k_3 + 4k_4 = -12$

$-14k_3 = -72$

$\boxed{k_3 = 5,14}$

$k_4 = \frac{-12 - 5,14}{4} = 4,3$ $\boxed{k_4 = 4,3}$

$k_2 = \frac{-12 - 5,14}{4} = 4,3$ $\boxed{k_2 = 4,3}$

$f_{1,2}(x) = \frac{-4,3}{6}\left[\frac{(x-1)^3}{-1} - (x-1)(-1)\right] - \frac{1\,(x-1)}{-1}$

$f_{1,2}(x) = \frac{-4,3}{6}\left[-(x-1)^3 + x - 1\right] + x - 1$

$f_{1,2}(1,5) = \frac{-4,3}{6}\left[-(0,5)^3 + 0,5\right] + 0,5$

$f_{1,2}(1,5) = \frac{-4,3}{6}(-0,125 + 0,5) + 0,5$

$f_{1,2}(1,5) = \frac{-4,3}{6} \times 0,375 + 0,5$

$\boxed{f_{1,2}(1,5) = 0,231}$

4) $M(x) = a + \frac{b}{x}$ (Sabit maliyet + x ile ters orantılı maliyet)

$$\text{Hata} = \sum_{i=1}^{3} \left(y_i - \left(a + \frac{b}{x_i}\right)\right)^2$$

$$= \left(110 - \left(a + \frac{b}{20}\right)\right)^2 + \left(60 - \left(a + \frac{b}{200}\right)\right)^2 + \left(50 - \left(a + \frac{b}{250}\right)\right)^2$$

a'ya göre türev $\cancel{-2}\left(110 - \left(a + \frac{b}{20}\right)\right) \cancel{-2}\left(60 - \left(a + \frac{b}{200}\right)\right) \cancel{-2}\left(50 - \left(a + \frac{b}{250}\right)\right) = 0$

$$-3a + b\left(-\frac{1}{20} - \frac{1}{200} - \frac{1}{250}\right) = -220$$

$$\Rightarrow \boxed{3a + \frac{59}{1000} b = 220}$$

b'ye göre türev $-\frac{2}{20}\left(110 - \left(a + \frac{b}{20}\right)\right) - \frac{2}{200}\left(60 - \left(a + \frac{b}{200}\right)\right) - \frac{2}{250}\left(50 - \left(a + \frac{b}{250}\right)\right) = 0$

$$5500 - 50a + \frac{50b}{20} + 300 - 5a - \frac{5b}{200} + 200 - 4a - \frac{4b}{250} = 0$$

$$\boxed{59a + \frac{2541b}{1000} = 6000}$$

$a \cong 49{,}5 \quad b \cong 1212$

$$M(x) = 49.5 + \frac{1212}{x}$$

100 adet için birim maliyet $\cong 49.5 + \frac{1212}{100} \cong 61.6$