Ödev 5, Çözümler

1. $y' = x + y \quad y(0) = 1$

$$y(x+h) \simeq y(x) + h y'(x) + \frac{h^2}{2} y''(x) \qquad y' = x + y \quad y'' = 1 + y' = 1 + x + y$$

$$y(x+h) \simeq y(x) + h.(x + y(x)) + \frac{h^2}{2}(1 + x + y(x))$$

$h = 0.1 \quad x = 0$

$$\Rightarrow y(0.1) = y(0) + (0.1)(0 + y(0)) + \frac{0.01}{2}(1 + 0 + y(0))$$

$$= 1 + 0.1 + 0.01 = 1.11$$

$h = 0.1 \quad x = 0.1$

$$y(0.2) = y(0.1) + 0.1.(0.1 + y(0.1)) + \frac{0.01}{2}(1 + 0.1 + y(0.1))$$

$$= 1.11 + (0.1)(1.21) + \frac{(0.01)(2.21)}{2}$$

$$= 1.11 + 0.121 + 0.01105$$

$$\simeq 1.23$$

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2. $y' = \underbrace{x+y}_{F(x,y)} \quad y(0) = 1$ Runge Kutta 2. derece $\Rightarrow$ $K_1 = hF(x,y)$

$K_2 = hF(x + \frac{h}{2}, y + \frac{K_1}{2})$

$y(x+h) = y(x) + K_2$

$x = 0 \quad h = 0.1$

$$K_1 = 0.1.(x + y(x)) = (0.1)(1) = 0.1$$

$$K_2 = 0.1(0.05 + 1.05) = 0.11$$

$$y(0.1) = 1 + 0.11 = 1.11$$

$x = 0.1 \quad h = 0.1$

$$K_1 = 0.1(0.1 + 1.11) = 0.121$$

$$K_2 = 0.1\left(0.15 + 1.11 + \frac{0.121}{2}\right) \simeq 0.1(1.32) = 0.132$$

$$y(0.2) = 1.11 + 0.132 = 1.242$$

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3. $y'' = x + y^2 + y' \quad y(0) = 1 \quad y'(0) = 0$

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix} \Rightarrow \vec{y}\,' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y_2 \\ x + y_1^2 + y_2 \end{bmatrix} \quad \vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{matrix} \to y_1(0) \\ \to y_2(0) \end{matrix}$$

$$\boxed{\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_2 \\ x + y_1^2 + y_2 \end{bmatrix} \quad \vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \to \text{1. derece vektörel form}$$

3. devam

Taylor serisi ilk iki terim $\Rightarrow \vec{y}(x+h) \cong \vec{y}(x) + h\vec{y}'(x)$

$$\vec{y}(x+h) \cong \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + h \begin{bmatrix} y_2 \\ x+y_1^2+y_2 \end{bmatrix} \qquad \begin{matrix} y_1 = y \\ y_2 = y' \end{matrix}$$

$x=0 \quad h=0.1$

$$\vec{y}(0.1) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} + 0.1 \begin{bmatrix} y_2(0) \\ 0+y_1(0)^2+y_2(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.1 \begin{bmatrix} 0 \\ 0+1^2+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}$$

$x=0.1 \quad h=0.1$

$$\vec{y}(0.2) = \begin{bmatrix} y_1(0.1) \\ y_2(0.1) \end{bmatrix} + 0.1 \begin{bmatrix} y_2(0.1) \\ 0.1+y_1(0.1)^2+y_2(0.1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0.1 \end{bmatrix} + \begin{bmatrix} 0.01 \\ (0.1)(1.2) \end{bmatrix} = \begin{bmatrix} 1.01 \\ 0.22 \end{bmatrix}$$

$y(0.2) = ? \quad \vec{y}_1(0.2) = ? \quad \Rightarrow \underline{y(0.2) = 1.01}$

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4. Tahmin yöntemi $\quad y'' = x+y^2+y' \quad y(0)=1 \quad y(1)=0$

$$\boxed{y'(0)=\alpha}$$

$$\Rightarrow \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \vec{y}' = \begin{bmatrix} y_2 \\ x+y_1^2+y_2 \end{bmatrix} \quad y_1 = y \quad y_2 = y'$$

$$\vec{y}(x+h) \cong \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + h \begin{bmatrix} y_2 \\ x+y_1^2+y_2 \end{bmatrix}$$

$\Rightarrow$ 3. soruda bunları çıkarmıştık

$x=0 \quad h=1/3$

$$\vec{y}(1/3) \cong \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} + \frac{1}{3} \begin{bmatrix} y_2(0) \\ 0+y_1(0)^2+y_2(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \alpha \end{bmatrix} + \frac{1}{3} \begin{bmatrix} \alpha \\ 0+1^2+\alpha \end{bmatrix} = \begin{bmatrix} 1+\frac{\alpha}{3} \\ \alpha + \frac{\alpha+1}{3} \end{bmatrix} = \begin{bmatrix} \frac{\alpha+3}{3} \\ \frac{4\alpha+1}{3} \end{bmatrix}$$

4. devam

$x=\frac{1}{3} \quad h=\frac{1}{3}$

$$\vec{y}\left(\frac{2}{3}\right)=\begin{bmatrix} y_1(\frac{1}{3}) \\ y_2(\frac{1}{3}) \end{bmatrix}+\frac{1}{3}\begin{bmatrix} y_2(\frac{1}{3}) \\ \frac{1}{3}+y_1(\frac{1}{3})^2+y_2(\frac{1}{3}) \end{bmatrix}$$

$$\vec{y}\left(\frac{2}{3}\right)=\begin{bmatrix} \frac{\alpha+3}{3} \\ \frac{4\alpha+1}{3} \end{bmatrix}+\frac{1}{3}\begin{bmatrix} \frac{4\alpha+1}{3} \\ \frac{1}{3}+\left(\frac{\alpha+3}{3}\right)^2+\frac{4\alpha+1}{3} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{\alpha+3}{3}+\frac{4\alpha+1}{9} \\ \frac{4\alpha+1}{3}+\frac{\alpha^2+18\alpha+13}{9} \end{bmatrix}=\begin{bmatrix} \frac{7\alpha+10}{9} \\ \frac{\alpha^2+30\alpha+16}{9} \end{bmatrix}$$

$$\rightarrow \frac{1}{3}+\frac{\alpha^2+6\alpha+9}{9}+\frac{(4\alpha+1).3}{3.3}$$

$$\frac{\alpha^2+18\alpha+13}{9}$$

$x=\frac{2}{3}, \quad h=\frac{1}{3}$

$$\vec{y}(1)=\begin{bmatrix} y_1(2/3) \\ y_2(2/3) \end{bmatrix}+\frac{1}{3}\begin{bmatrix} y_2(2/3) \\ \frac{2}{3}+y_1(\frac{2}{3})^2+y_2(2/3) \end{bmatrix}$$

$$=\begin{bmatrix} \frac{7\alpha+10}{9} \\ \frac{\alpha^2+30\alpha+16}{9} \end{bmatrix}+\frac{1}{3}\begin{bmatrix} \frac{\alpha^2+30\alpha+16}{9} \\ \frac{2}{3}+\frac{49\alpha^2+100+140\alpha}{81}+\frac{9\alpha^2+270\alpha+144}{81} \end{bmatrix}$$

$$\Rightarrow y(1)=\frac{7\alpha+10}{9}+\frac{\alpha^2+30\alpha+16}{27}=\frac{\alpha^2+51\alpha+46}{27}=0 \rightarrow y(1) \text{ soruda verilen}$$

$\Rightarrow \alpha$ bulunur $\quad \alpha=-0.92$

$$\Rightarrow y(1/3)\simeq\frac{\alpha+3}{3}\simeq 0.69 \,//$$

$$y(2/3)\simeq\frac{7\alpha+10}{9}\simeq 0.396 \,//$$

5. $y'' = x + y^2 + y'$ $\quad y(0) = 1 \quad y(1) = 0$

$$\frac{y(x+h) - 2y(x) + y(x-h)}{h^2} = x + y(x)^2 + \frac{y(x+h) - y(x-h)}{2h}$$

$$\boxed{x = \frac{1}{3},\ h = \frac{1}{3}} \Rightarrow \underbrace{y\left(\tfrac{2}{3}\right)}_{y_2} - 2\underbrace{y\left(\tfrac{1}{3}\right)}_{y_1} + \underbrace{y(0)}_{1} = \frac{1}{9}\left(\frac{1}{3} + y\left(\tfrac{1}{3}\right)^2 + \frac{y\left(\frac{2}{3}\right) - y(0)}{\frac{2}{3}}\right)$$

$$\Rightarrow y_2 - 2y_1 + 1 = \frac{1}{27} + \frac{y_1^2}{9} + \frac{y_2 - 1}{6} \Rightarrow \boxed{6y_1^2 + 108y_1 - 45y_2 - 61 = 0}$$

$$\boxed{x = \frac{2}{3},\ h = \frac{1}{3}} \quad \underbrace{y(1)}_{0} - 2y_2 + y_1 = \frac{1}{9}\left(\frac{2}{3} + y_2^2 + \frac{y(1) - y_1}{\frac{2}{3}}\right)$$

$$\Rightarrow \boxed{6y_2^2 + 108y_2 - 63y_1 + 4 = 0}$$

Bu iki denklemi çözmek için Newton Raphson kullanılır

$$\vec{f} = \begin{bmatrix} 6y_2^2 + 108y_2 - 63y_1 + 4 \\ 6y_1^2 + 108y_1 - 45y_2 - 61 \end{bmatrix} \qquad J = \begin{bmatrix} -63 & 12y_2 + 108 \\ 12y_1 + 108 & -45 \end{bmatrix}$$

$$\vec{y}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Delta x = -J^{-1}\vec{f}$$

$$\vec{y}^{(1)} = \vec{y}^{(0)} + \vec{\Delta x}$$

matlab ile

3 iterasyon sonra

$y_1 = 0.6872$

$y_2 = 0.7568$