Outline

- Image Restoration
  - Introduction
  - Mathematical modeling
  - Noise models
  - Denoising in the spatial domain
  - Denoising in the frequency domain

Image Restoration: Introduction

- Goal: Recover an image that is altered by a system
- Most of the time, the system itself is known, or some prior information is available
- If the degradation is known, a form of inverse is applied to the observed image to estimate the desired image
- If the degradation is not known, the degradation and the image is estimated simultaneously, such techniques are called “blind” estimation techniques

Image Restoration: Introduction (Cont.)

- Examples:
  - An image is blurred, and we want to recover the original image by deblurring
  - The image is altered significantly due to the physics of the image acquisiton technique, e.g. PET image reconstruction where the measurements are not the images themselves but the image altered by a linear system
  - Series of images (video) are acquired when the object or the camera is moving, blurring occurs, but it is not a simple blurring depending on the motion path of the camera or the object
  - An image is acquired from an angle resulting e.g. perspective deformation, we would like to recover the original image

Image Restoration: Mathematical Modeling

- Assuming that the degradation function is linear shift invariant we have the observation model

$$g(x, y) = h(x, y) * f(x, y) + n(x, y)$$

- In the frequency domain

$$G(u, v) = H(u, v) * F(u, v) + N(u, v)$$

- Then the estimate of the original function is given by a restoration system $r(x, y)$

$$\hat{f}(x, y) = r(x, y) * g(x, y)$$

- Or in the frequency domain

$$\hat{F}(u, v) = R(u, v)G(u, v)$$

Image Restoration: Mathematical Modeling (Cont.)

- The restoration function $r(x, y)$ is selected depending on a cost function

$$f_{\text{opt}}(x, y) = \arg \min_{f(x,y)} \{\hat{g}(x, y), r(x, y), f(x, y)\}$$

- If we are considering the blind techniques then we have

$$f_{\text{opt}}(x, y) = \arg \min_{f(x,y), r(x,y)} \{\hat{g}(x, y), r(x, y), f(x, y)\}$$

Image Restoration: Noise Models

- Noise model that best fits the restoration problem depends very much on how the image is acquired
- For simplicity, we assume that the noise is spatially invariant and independent from the image

Image Restoration: Noise Models (Cont.)

- Noise Models
  - Gaussian noise: can be valid in several applications due to the central limit theorem

$$p(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(z-\mu)^2/2\sigma^2}$$

    where μ denotes the mean and σ^2 denotes the variance
  - Rayleigh noise: useful for modeling non-symmetric noise

$$p(z) = \begin{cases} \frac{2}{b}(z-a)e^{-(z-a)^2/b} & z \geq a \\ 0, & \text{otherwise} \end{cases}$$

    resulting in the mean and variance values

$$\mu = a + \sqrt{\pi b/4} \qquad \text{and} \qquad \sigma^2 = \frac{b(4-\pi)}{4}$$

Image Restoration: Noise Models (Cont.)

- Exponential noise: useful for positive noise values when large noise values are less probable

$$p(z) = \begin{cases} ae^{-az} & z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- Impulse Noise (Salt and Pepper): models point noises

$$p(z) = \begin{cases} P_a & z = a \\ P_b & z = b \\ 0 & \text{otherwise} \end{cases}$$

- Poisson Noise: useful when the image acquisition have discrete counts (such as photon or electrons)

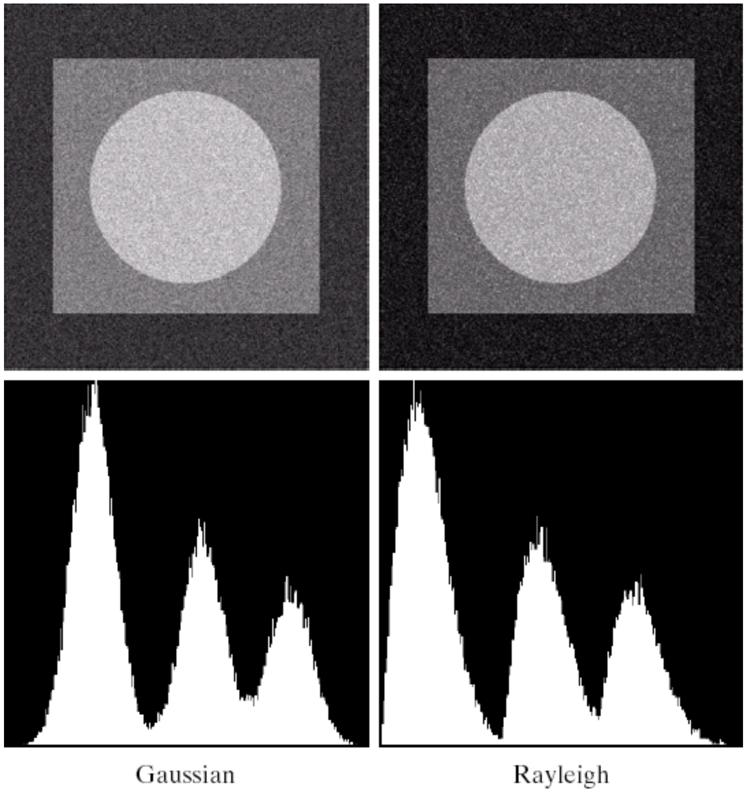
$$p(z; \lambda) = \frac{e^{-\lambda}\lambda^z}{z!}$$

with mean and variance both equal to λ

Image Restoration: Noise Models (Cont.)

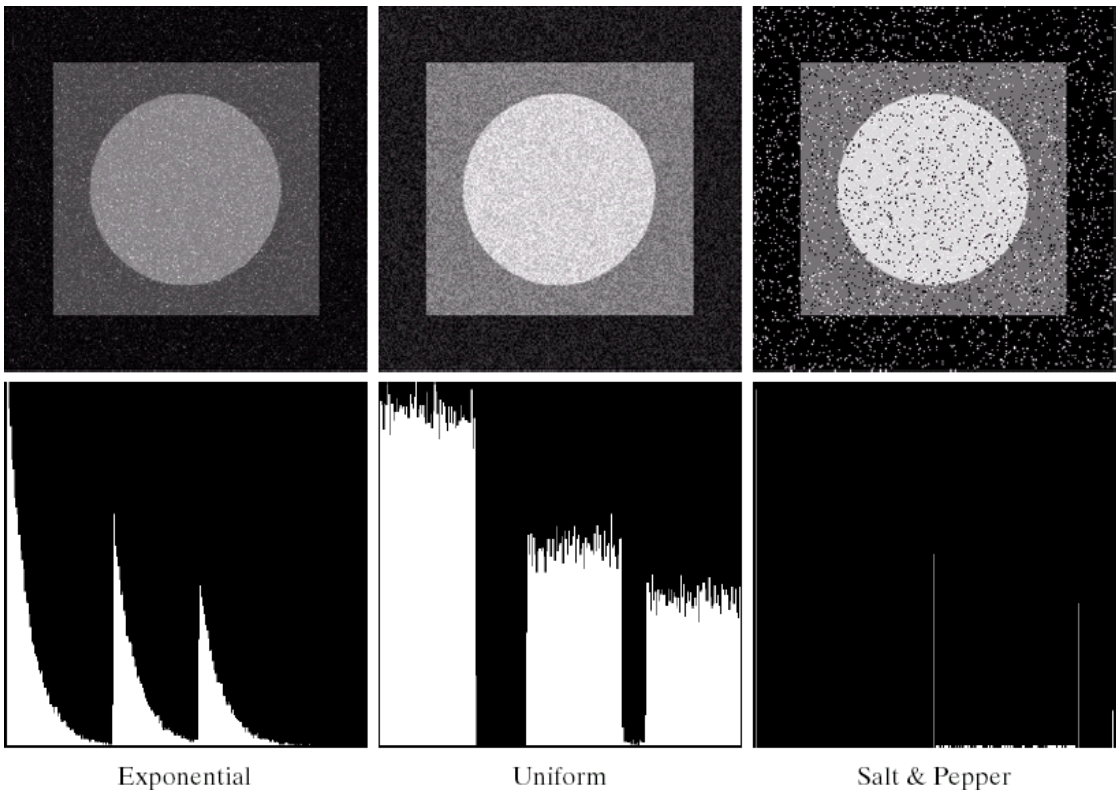
- Periodic noise: watch out! spatially variant noise, useful for modeling noise due to electronics
- Estimation of noise
  - Some prior information or the physics of the image acquisition technique can be used to model the noise
  - A known simple object can be imaged, and the resulting image will have information related to the system noise

Image Restoration: Estimation of Noise Parameters



Left: Normal noise, right: Rayleigh noise

Image Restoration: Estimation of Noise Parameters (Cont.)



Left: Exponential noise, middle: uniform noise, salt and pepper noise

Image Restoration: Estimation of Noise Parameters (Cont.)

- Once the constant object is imaged, estimates of the mean and variance can be obtained by

$$\hat{\mu} = \sum_{z_i} z_i p(z_i) \quad \text{and} \quad \hat{\sigma}^2 = \sum_{z_i} (z_i - \mu)^2 p(z_i)$$

- Mean and variance can be used to solve for the unknown parameters of particular noise PDF's

Image Restoration: Noise Reduction with Mean Filters

- In this case the noise is the only degradation, that is $h(x, y)$ is the identity operation

$$g(x, y) = f(x, y) + n(x, y)$$

$$G(u, v) = F(u, v) + N(u, v)$$

- We can use spatial filters to reduce noise as noted in the previous lectures
  - Arithmetic mean: $\hat{f}(x, y) = \frac{1}{mn} \sum_{s,t} g(s, t)$
  - Geometric mean: $\hat{f}(x, y) = [\sum_{s,t} g(s, t)]^{\frac{1}{mn}}$
  - Harmonic mean: $\hat{f}(x, y) = \frac{mn}{\sum_{st} \frac{1}{g(s,t)}}$
  - Contraharmonic mean filter, more general

$$\hat{f}(x, y) = \frac{\sum_{s,t} g(s, t)^{Q+1}}{\sum_{s,t} g(s, t)^{Q}}$$

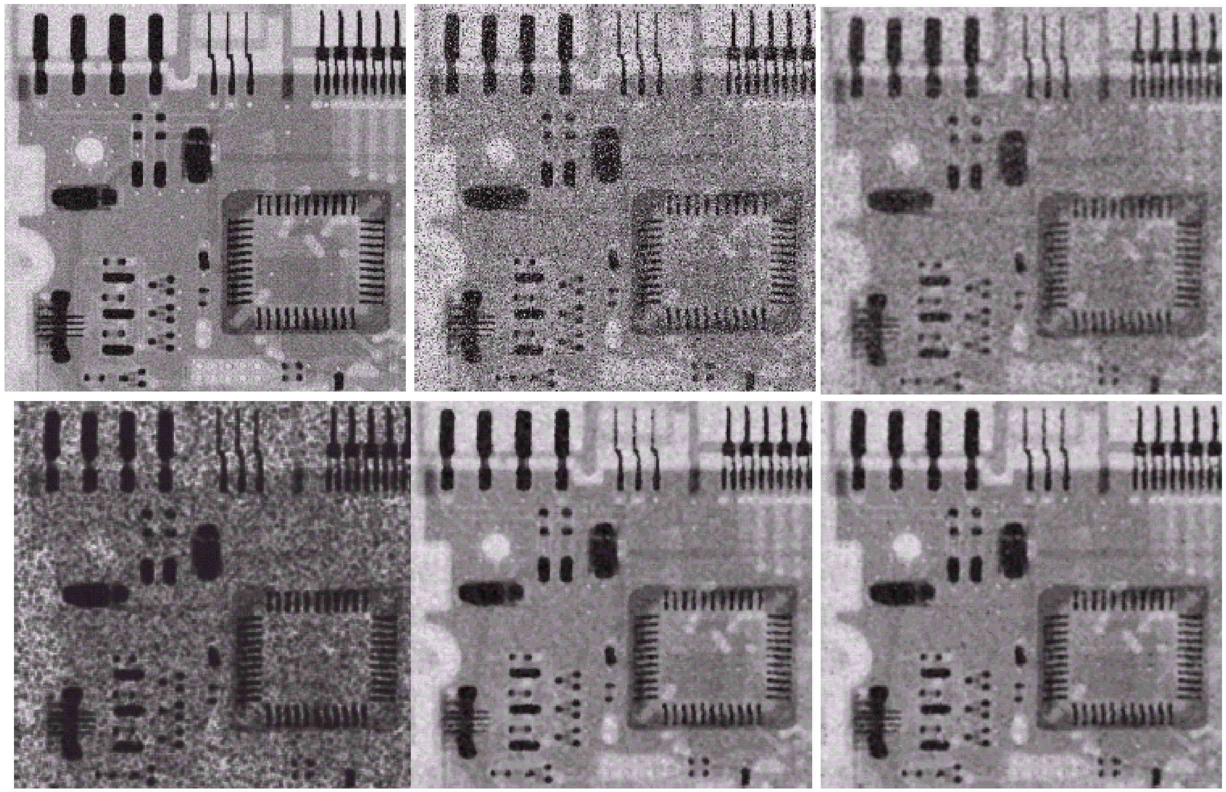
  - Contraharmonic reduces to arithmetic when $Q = 1$ and harmonic when $Q = -1$

Image Restoration: Noise Reduction with Order-Statistics Filters

- Median: $\hat{f}(x, y) = \text{median}\{f(x, y)\}$
- Max: $\hat{f}(x, y) = \max\{f(x, y)\}$
- Min: $\hat{f}(x, y) = \min\{f(x, y)\}$
- Midpoint: $\hat{f}(x, y) = \{\min\{f(x, y)\} + \max\{f(x, y)\}\}/2$
- Trimmed filter: zero $d/2$ lowest and $d/2$ highest values in the window producing $g_d(s, t)$, then trimmed filter results in

$$\hat{f}(x, y) = \frac{1}{mn - d} \sum_{s,t} g_d(s, t)$$

Image Restoration: Noise Reduction Image Example



Top-left: additive noisy image, Top-middle: salt and pepper noise added, top-right: arithmetic filter, bottom-left: geometric filter, bottom-middle: median filter, bottom-right: trimmed filter, $d = 5$

Image Restoration: Adaptive Filters

- Until now, all filters are assumed to be global, that is same value regardless of the image content
- Adaptive filters are filters with varying characteristic with respect to the image content
- Improved performance, but inreased cost since the filter coefficients vary
- Let our filter to vary with respect to the mean and variance of the image window that is being filtered
- One way to do adaptive filtering is to use a lot of smoothing when the local variance is low (no detailed information) to reduce noise, and use filtering close to identity when the variance is high (detailed information that we want to keep)

Image Restoration: Adaptive Filters (Cont.)

- A filter that performs these are

$$\hat{f}(x,y) = g(x,y) - \frac{\sigma_n^2}{\sigma_L^2}[g(x,y) - \mu_L]$$

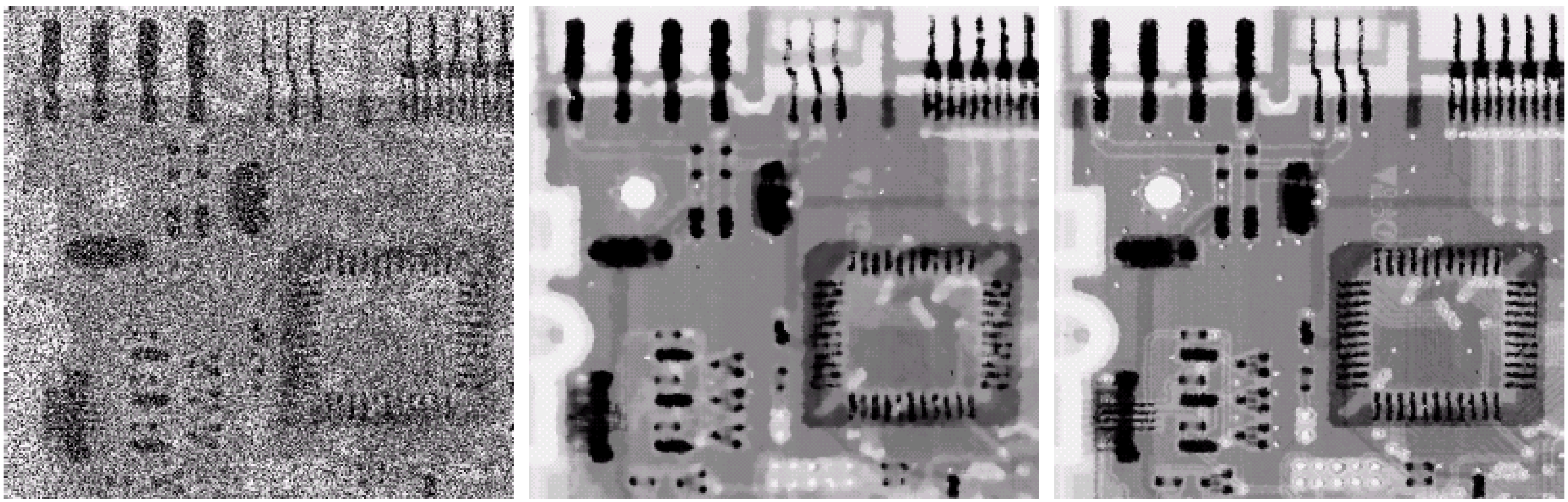
  where σ_n^2 is the global image variance, σ_L^2 the local image variance, and μ_L the local mean

- Assumes that $\sigma_n^2 \leq \sigma_L^2$, this should be checked, if violated set it to 1, or arrange dynamic range so aht resulting negative gray values are corrected

Image Restoration: Adaptive Filters (Cont.)

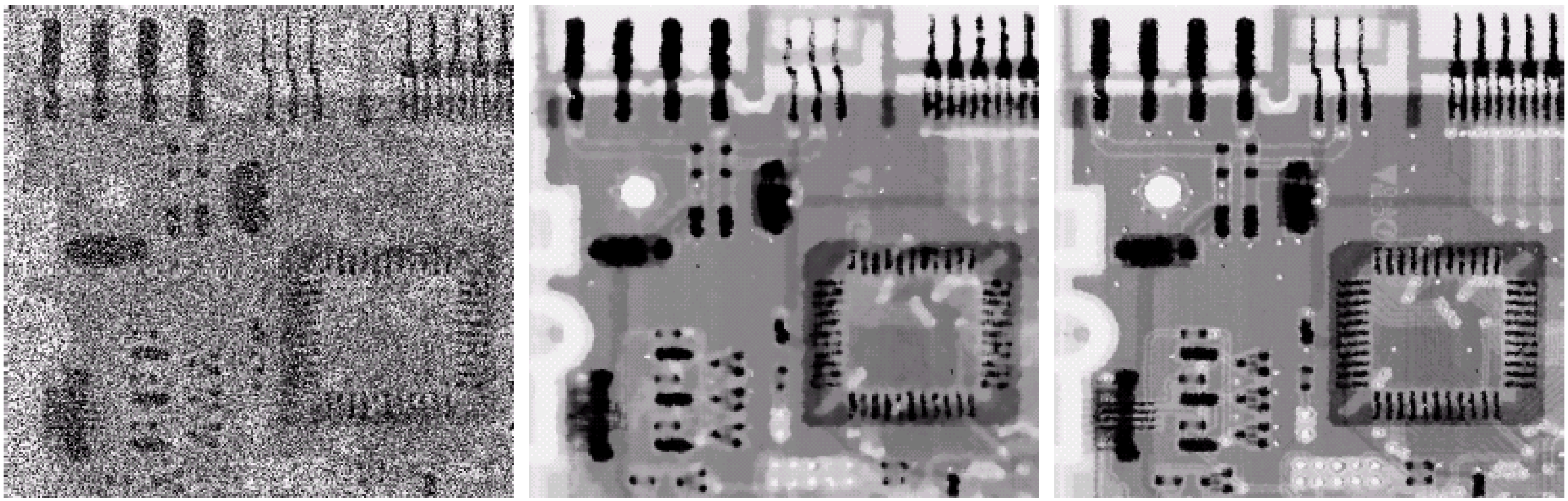
- We can have an adaptive median filter that changes size with respect to the image content, larger size $\rightarrow$ more blurring, smaller size $\rightarrow$ less blurring
- The steps of adaptive median filtering is as follows
  - If the median is not between the min and max increase window size, until it is or, until a maximum window size is reached
  - Then if the current pixel is not the min or max, keep it; otherwise replace it with the median
  - This algorithm provides the reduction of salt-pepper noise, but also reduces blurring preserving detailed information which migh be valuable

Image Restoration: Adaptive Filters Image Example



Top-left: additive noisy image, Top-right: arithmetic mean, bottom-left: geometric mean, bottom-right: adaptive filter

Image Restoration: Adaptive Median Filter Image Example



Left: noisy image, middle: median, right: adaptive median

Image Restoration: Image Denoising in the Frequency Domain

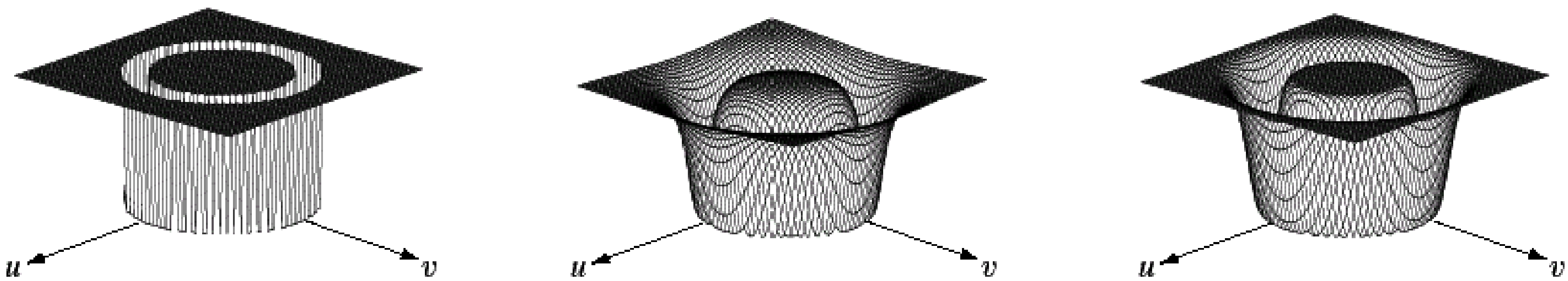
- We add a few more filters to the ones discussed before for noise reduction
- Bandpass filters: we consider similar (ideal, Butterworth, and Gaussian bandpass filters which are given by

$$H_{\mathrm{I}} = \begin{cases} 0 & D_0 - W/2 \leq D(u,v) \leq D_0 + W/2 \\ 1 & \text{otherwise} \end{cases}$$

$$H_{\mathrm{BW}} = \frac{1}{1 + \left[\frac{D(u,v)W}{D^2(u,v) - D_0^2}\right]^{2n}}$$

$$H_{\mathrm{G}} = 1 - e^{-\frac{1}{2}\left[\frac{D^2(u,v) - D_0^2}{D(u,v)W}\right]^{2n}}$$

Image Restoration: Image Denoising in the Frequency Domain (Cont.)



Left: ideal, middle: Butterworth, right: Gaussian

Image Restoration: Notch Filters

- We can similarly have bandreject filters which are simply

$$H_{\mathrm{BR}}(u,v) = 1 - H_{\mathrm{BP}}(u,v)$$

- Notch filters are filters that pass/reject frequencies around one o rmore frequency values
- An ideal notch filter is

$$H_{\mathrm{N}}(u,v) = \begin{cases} 0 & D_1(u,v) \leq D_0 \quad \text{or} \quad D_2(u,v) \leq D_0 \\ 1 & \text{otherwise} \end{cases}$$

where

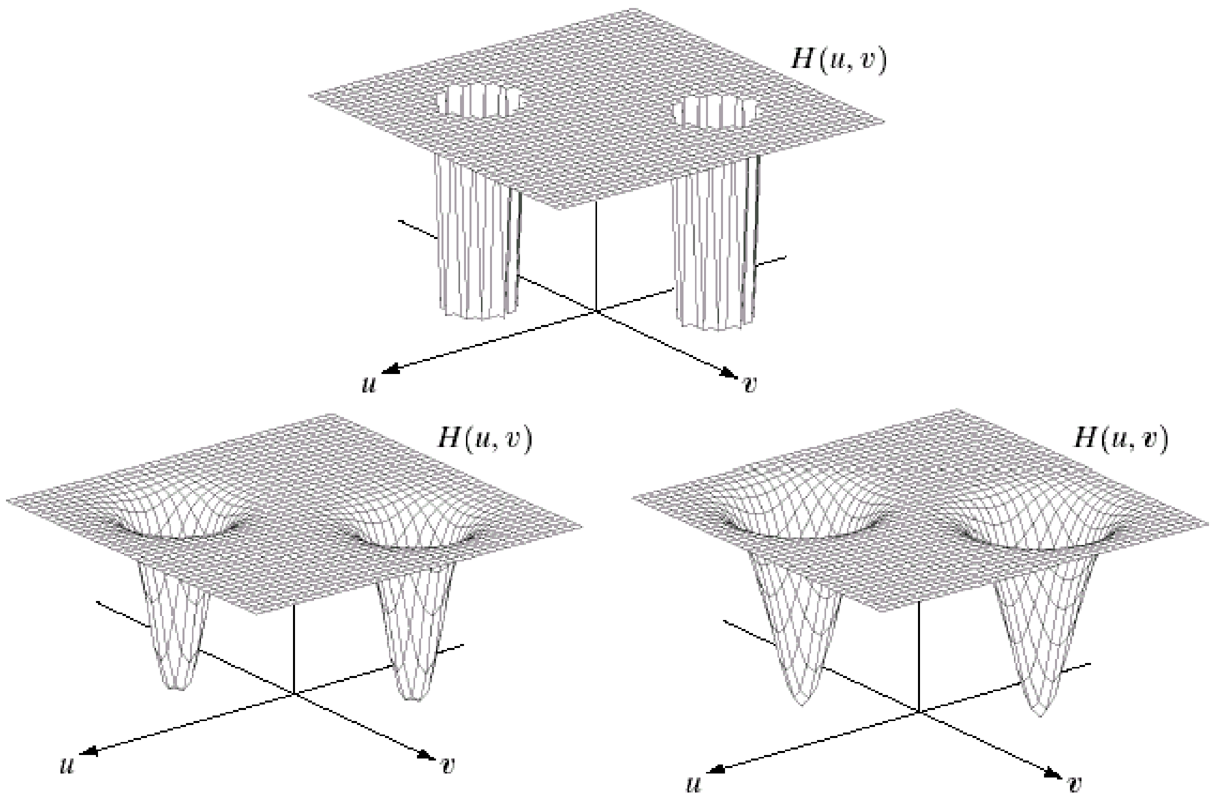
$$D_1(u,v) = [(u - M/2 - u_0)^2 + (v - N/2 - v_0)^2]^{1/2}$$

and

$$D_1(u,v) = [(u - M/2 + u_0)^2 + (v - N/2 + v_0)^2]^{1/2}$$

- Other notch filters can be derived based on the Butterworth and Gaussian

Image Restoration: Notch Filters(Cont.)



Top: ideal, bottom-right: Butterworth, bottom-left: Gaussian

Image Restoration: Optimum Notch Filters

- In several cases there will be more than one noise components scattered in the frequency domain
- We extract these noise components by a notch filter and then take the inverse FT to obtain the noise in spatial domain
- Denoised image can be obtained by subtracting noise from the corrupted image
- Mathematically

$$\hat{f}(x, y) = g(x, y) - w(x, y)n(x, y)$$

where $w(x, y)$ is a weighting function and

$$n(x, y) = \mathcal{F}^{-1}\{H(u, v)G(u, v)\}$$

with $H(u, v)$ the notch filter and $G(u, v)$ the corrupted image

Image Restoration: Optimum Notch Filters (Cont.)

- "Optimum" notch filters that are the ones minimizing the local variance
- Local variance is

$$\sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_s \sum_t [\hat{f}(x+s, y+t) - \bar{\hat{f}}(x,y)]^2$$

  where

$$\bar{\hat{f}}(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_s \sum_t \hat{f}(x+s, y+t)$$

- Asssuming a locally constant weight function, and taking the derivative with respect to $w(x,y)$, equating to zero results in

$$w(x,y) = \frac{\overline{g(x,y)n(x,y)} - \bar{g}(x,y)\bar{n}(x,y)}{\overline{n^2}(x,y) - \bar{n}^2(x,y)}$$

- Calculate the optimum weight using these equations and use the notch filter with this weight