Outline

- Image Restoration: Linear Shift Invariant Degradations
- Estimation of Linear Shift Invariant Degradations
- Image Restoration by Inverse Filtering
  - Minimum Mean Square Error Filtering
  - Constrained Least Squares Filtering
- Image Restoration by Geometric Transformations
- Multiresolution Image Processing: Introduction
- Multiresolution Image Processing: Image Pyramids
- Multiresolution Image Processing: Subband Coding and Filterbanks
- Practice Problems of Image Restoration

Image Restoration: Linear Shift Invariant Degradations

- Let us assume that the degradation function is shift invariant then, the degradation model becomes

$$g(x,y) = \int\int f(x,y,x',y')h(x-x',y-y')\mathrm{d}x'\mathrm{d}y' + n(x,y)$$

whereas it is

$$g(x,y) = \int\int f(x,y,x',y')h(x,x',y,y')\mathrm{d}x'\mathrm{d}y' + n(x,y)$$

for a general linear degradation

- The degradation function is 4-D for the general linear case while it is 2-D for the shift-invariant case

- The degradation model can be written in the Fourier domain *only* when the degradation is shift invariant

$$G(u,v) = F(u,v)H(u,v) + N(u,v)$$

Estimation of the Degradation Function

- We can estimate the degradation function in several ways depending on the prior knowledge we have and acessability of the imaging system
  - Estimation by image observation: assume that we know the characteristics of the object and we can reconstruct the original image by e.g. deblurring
  - Estimation by experimentation: assuming the imaging equipment is available for experimentation
  - Parametric modeling: assuming some prior knowledge on the system

Image Restoration: Esimation of the Degradation Function by Image Observation

- Assume that we can obtain a good estimate of the original image by some technique. Then the degradation function for a part of the image with high SNR is

$$H_s(u,v) = \frac{G_s(u,v)}{\hat{F}_s(u,v)}$$

  Since we assume LSI degradation, we can construct the $H(u,v)$ for the whole image using $H_s(u,v)$

Image Restoration: Estimation of the Degradation Function by Experimentation

- If we have access to the imaging system, then we can experiment with it until we obtain similar degradation of the images at hand
- Once the imaging sytem is set to operata on a set of conditions, we can image an impulse response, than the resulting image will simply be the degradation of the sytem

$$H(u,v) = G(u,v)/A$$

where $G(u,v)$ is the image of an impulse with power A

Image Restoration: Estimation of the Degradation Function by Parametric Modeling

- Assuming that we have prior knowledge related to the degradation function (or the imaging system), we can use this knowledge to construct a parametric model, and then estimate only these parameters instead of estimating the whole degradation function
- We model the degradation function as

$$H(u, v) = D(u, v, \theta_1, \ldots, \theta_m)$$

  where D is a known functional form, and $\theta_1, \ldots, \theta_m$ are the unknown parameters of the degradation function that need to be estimated
- Most of the time D can be obtained using the physics of the imaging system

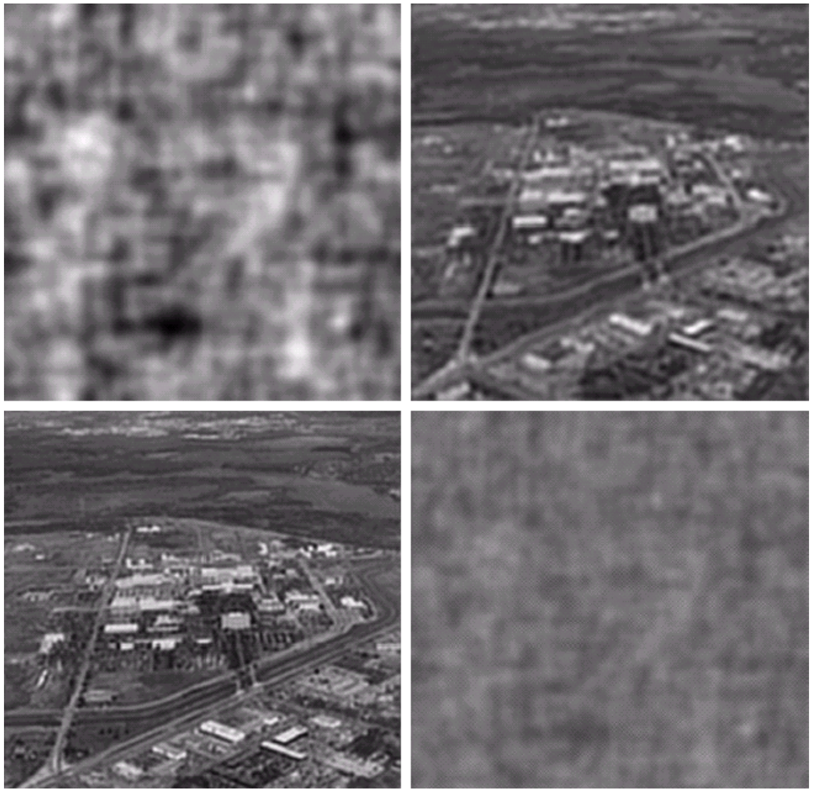
Image Restoration by Inverse Filtering

- We can invert the degradation function and apply it to the observation to recover an estimate of the original image

$$\hat{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

- The problem is that the noise values are amplified when $H(u,v)$ has a small value
- This problem is called as being "ill-conditioned"
- One solution is simply exclude the values of $H(u,v)$ which are very small by considering the low pass region

Image Restoration by direct Inverse Filtering: Image Example



Top-left: Observed, top-right: direct inverse filtering with radius 40, bottom-left: radius 70, bottom-right: radius 85

Image Restoration by Minimum Mean Squared Error Filtering

- Since direct inverse filtering have problems of amplifying the noise, let us consider an inverting scheme that takes noise into account
- We will choose a filter that minimizes the mean squared error (MSE) between the estimated and the original image

$$\hat{f}_{\mathrm{MMSE}} = \arg\min_{\hat{f}} \mathrm{E}\{|f - \hat{f}|^2\}$$

- The solution can be obtained by taking derivatives and equating to zero

$$\hat{F}(u,v) = \left[\frac{H^*(u,v) S_f(u,v)}{S_f(u,v)|H(u,v)|^2 + S_n(u,v)} \right] G(u,v)$$

Image Restoration by Minimum Mean Squared Error Filtering (Cont.)

- Rearranging terms

$$\hat{F}(u,v) = \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + S_n(u,v)/S_f(u,v)}\right]$$

- Note that the Wiener Filter requires the knowledge of second order statistics of the image and noise, S_f and S_n, the power spectrums of the image and noise.

- If these are now known we can replace the ratio by a constant (related to SNR)

Image Restoration by Direct Inverse Filtering and Wiener Filtering: A Comparison

Left: Direct inverse filtering, center: radially limited direct inverse filtering, right: Wiener filtering

Image Restoration by Constrained Filtering

- Wiener filter requires knowledge of power spectrum, and approximations of constant is not always a good approximation
- Remember the problem was the noise amplification in direct filtering
- In constrained filtering, we basically minimize the noise variance (limiting noise amplification) subject to the constraint coming from the degradation model
- Mathematically

$$\hat{f} = \arg\min_{\hat{f}} \sum \sum [\nabla^2 f(x,y)]^2$$

subject to the constraint

$$|\boldsymbol{g} - \boldsymbol{H}\hat{\boldsymbol{f}}|^2 = |\boldsymbol{n}|^2$$

Image Restoration by Constrained Filtering (Cont.)

- The solution to this constrained problem is

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma |P(u,v)|^2} \right] G(u,v)$$

where $P(u,v)$ is the FT of

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & -4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

and γ is a free parameter to be adjusted to satisfy the constraint

Image Restoration by Constrained Filtering (Cont.)

- To calculate the γ value systematically using an iterative approach, we define

$$\boldsymbol{r} = \phi(\gamma) = |\boldsymbol{g} - \boldsymbol{H}\hat{\boldsymbol{f}}|^2$$

  which can be shown to be an increasing function of γ

- Simply start with an initial γ value, and then increase it or decrease it depending on $|R|^2$ being larger or smaller than $|n|^2$ within a tolerance range
- Stop when $|r|^2$ is close enough to $|n|^2$
- The quantities required are $|r|^2$ and $|n|^2$
- The term $|r|^2$ can be calculated by its definition

Image Restoration by Constrained Filtering (Cont.)

- The term $|n|^2$ can be calculated by using the estimate of the variance

$$\sigma_n^2 = \frac{1}{MN} \sum \sum [n(x,y) - \mu_n]^2$$

  where $\mu_n = 1/MN \sum \sum n(x,y)$

- Arranging terms results in

$$|n|^2 = MN[\sigma_n^2 + \mu_n^2]$$

- The algorithm then only requires the knowledge of mean and variance of the noise which can be estimated

Image Restoration by Geometric Transformations (Cont.)

- Spatial transformations of freq. domain filtering: modifies pixel values
- Geometric transformations: modifies pixel locations
- Examples of geometric transformations: perspective transformation, translation, rotation, affine transformations, etc.
- Assume the original coordinates of a pixel are x, y, then the transformed coordinates (x', y') are

$$x' = r(x, y) \qquad \text{and} \qquad y' = s(x, y)$$

  where r and s determine the type and nature of the geometric transformations
- Our goal is to find what the deformations (r and s) are and then apply the inverses to restore the original image

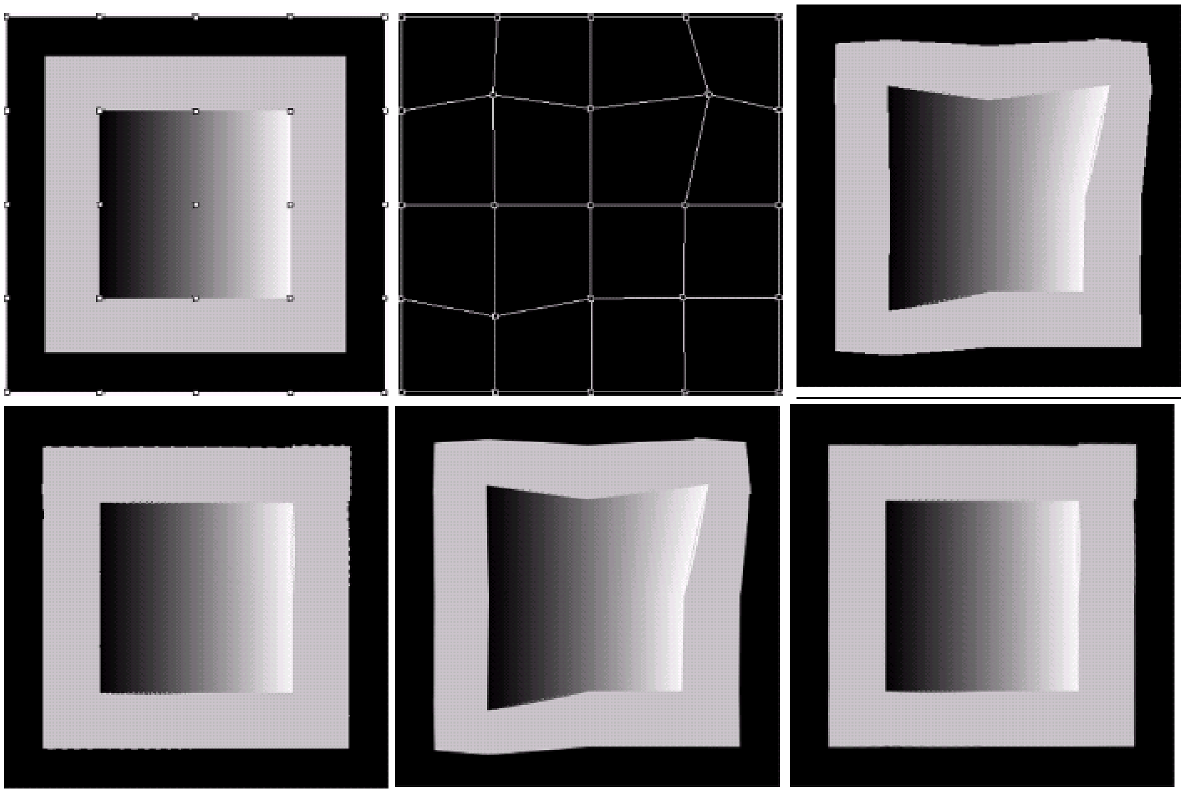
Image Restoration by Geometric Transformations: Landmarks

- In several applications, isolated points or features are used to estimate r and s
- These isolated points and features are called landmarks, and their locations are assumed to be known both at the original and the distorted image
- We need sufficient number of landmarks so that all parameters in r and s can be estimated
- E.g.: MR images, the patient or subject are attached visible marks with known exact location. When the MR image is produced, these locations will be varied because of deformation. Since we know the original locations of these landmarks, we can find the deformation function and apply the inverse to the whole image to restore the original image
- Also known as a subtopic of "image registration" where the goal is to align two images

Image Restoration by Geometric Transformations: Interpolation

- When s and r values are found, to restore the original image we almost always need values of the image where it is not sampled (non-integer pixel values)
- Several techniques can be used for interpolation considerin the tradeoff between complexity and accuracy
- Simplest method is to use the nearest neighbour, not so accurate but very simple
- More accurate methods such as bilinear or cubic interpolation methods can be used with the price of increased computation

Image Restoration by Geometric Transformations: Image Example



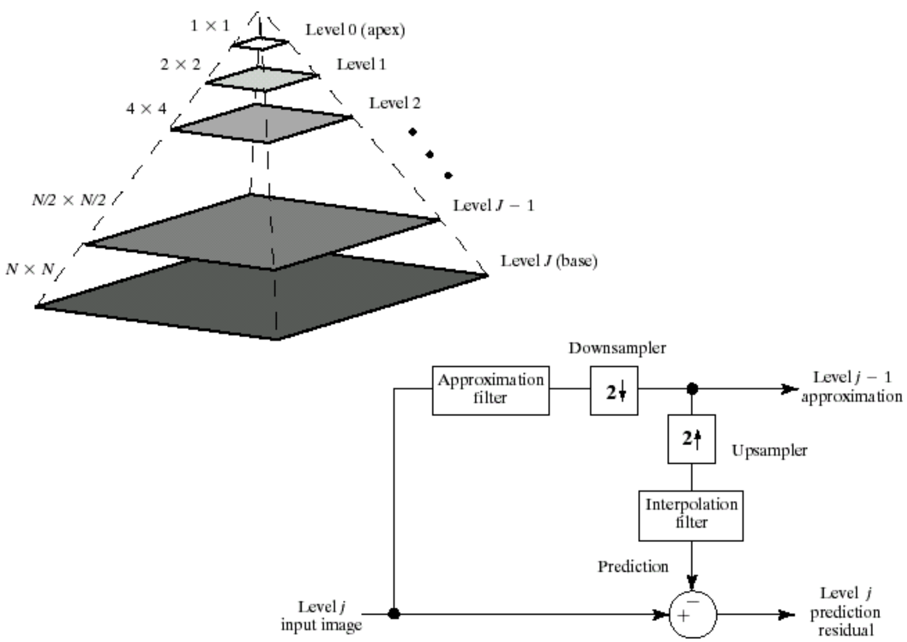
Top-left: original image, top-middle: distorted landmarks, top-right: distorted image (nearest neighbour), bottom-left: restored image (nearest neighbour), bottom-middle: distorted image (bilinear), bottom-left: restored image (bilinear)

Multiresolution Image Processing: Introduction

- Multi-resolution image processing processes the image at different resolutions
- Wavelet transform is a mathematical tool that allows us to perform multi-resolution processing
- Wavelet transform is a transform with basis functions other than exponentials (as in the FT)
- The basis functions are limited in space also, hence carrying information related to both frequency and space

Multiresolution Image Processing: Image Pyramids

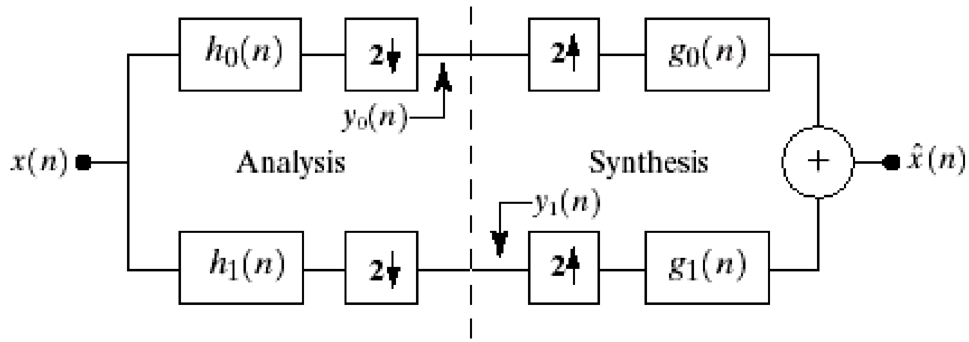
- An image pyramid is constructed by downsampling the image at each step by 2 until a 1×1 image is reached
- Both the coarse approximations and the differences between the original and coarse approximations can be stored so that the original image can be restored lates

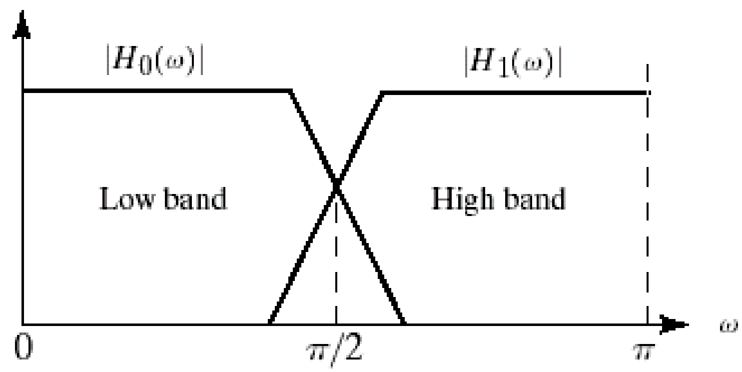


Left: Image pyramid, right: block diagram for one level

Multiresolution Image Processing: Subband coding

- We separate the original image into its subbands using filterbanks





Top: The filterbank, bottom: subbands of the image

Multiresolution Image Processing: Subband Coding

- The reconstruted image at the left handside of the filter bank has the z-transform

$$\begin{aligned} X(z) &= 0.5[H_0(z)G_0(z) + H_1(z)G_1(z)]X(z) \\ &\quad +0.5[H_0(-z)G_0(-z) + H_1(-z)G_1(-z)]X(-z) \end{aligned}$$

- If we would like to have a perfect reconstruction (PR, shifts and intensity scaling ok), then we must have

$$H_0(z)G_0(z) + H_1(z)G_1(z) = Kz^{-k}$$

$$H_0(-z)G_0(-z) + H_1(-z)G_1(-z) = 0$$

- We can have several solutions to these equations

Multiresolution Image Processing: Subband Coding (Cont.)

Filter	QMF	CQF	Orthonormal
$H_0(z)$	$H_0^2(z) - H_0^2(-z) = 2$	$H_0(z)H_0(z^{-1}) +$ $H_0^2(-z)H_0(-z^{-1}) = 2$	$G_0(z^{-1})$
$H_1(z)$	$H_0(-z)$	$z^{-1}H_0(-z^{-1})$	$G_1(z^{-1})$
$G_0(z)$	$H_0(z)$	$H_0(z^{-1})$	$G_0(z)G_0(z^{-1}) +$ $G_0(-z)G_0(-z^{-1}) = 2$
$G_1(z)$	$-H_0(-z)$	$zH_0(-z)$	$-z^{-2K+1}G_0(-z^{-1})$

Three PR filterbanks