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Multiresolution Expansions: General Series Expansions

- A signal can be represented as a linear combination of expansion functions

$$f(x) = \sum_k \alpha_k \phi_k(x)$$

- If the expansion function is complete, then any function can be represented using a discrete set of α_k values
- If the basis functions are orthonormal then α_k's can be calculated easily using

$$\alpha_k = \int \phi_k^*(x) f(x) \mathrm{d}x$$

- If the basis functions are orthogonal then we have

$$\alpha_k = \int \tilde{\phi}_k^*(x) f(x) \mathrm{d}x$$

where $\tilde{\phi}_k$ is a dual basis functions that are orthonormal to the original basis functions

Multiresolution Expansions

- We use scaling functions to create approximations of an image at different resolutions
- The difference between the original and the approximations can be also encoded and kept for perfect reconstruction
- A scaling function has the general form

$$\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k)$$

- The parameter k denotes the location, j the width and amplitude of the function
- If we choose an appropriate $\phi(x)$ we can create a basis functions that are complete and orthonormal

An Example of Multiresolution Expansions: Haar Function

- Consider the simplest expansion function $\phi(x)$

$$\phi(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Using this function and scaling functions we can create a set of basis functions that can represent any 2-D image

Multiresolution Expansions: MRA Requirements

- Scaling Functions is orthogonal to its integer translates
- The subspaces spanned at low scales (small j values) are within those spanned at higher scales (large j values)

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \ldots$$

- The zero image is in the span of all expansion functions at any scale

$$V_{-\infty} = 0$$

- Any L^2 can be represented with arbitrary precision with the basis functions when all scales are used

Multiresolution Expansions: MRA Requirements

- If all four MRA requirements are satisfied then

$$\phi_{jk}(x) = \sum_n \alpha_n \phi_{j+1,n}(x)$$

- Since ϕ is a scaling function we obtain

$$\phi_{jk}(x) = \sum_n h_\phi(n) 2^{(j+1)/2} \phi(2^{j+1}x - n)$$

- Or more simply written as

$$\phi(x) = \sum_n h_\phi(n) \sqrt{2} \phi(2x - n)$$

called the refinement or dilation equation

Multiresolution Expansions: Wavelet Functions

- Now let us consider a second set of functions called wavelet functions $\psi(x)$ that span the difference denoted by W_j between adjacent scaling subspaces

$$V_{j+1} = V_j \oplus W_j$$

- Wavelets then are defined as

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$$

- We can represent any function as part of the space

$$V_0 \oplus W_0 \oplus W_1 \oplus W_2 \ldots$$

- We can also eliminite V_0 by going to the negative values

$$\ldots W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \ldots$$

Multiresolution Expansions: Wavelet Functions (Cont.)

- Since wavelets are created with scaling functions

$$\psi(x) = \sum_n h_\psi(n)\sqrt{2}\psi(2x - n)$$

- The relation between ϕ and ψ is

$$h_\phi(n) = (-1)^n h_\psi(1 - n)$$

Haar Wavelets

- Using the haar functions and previous equation, we can obtain Haar wavelets

$$\psi(x) = \begin{cases} 1 & 0 < x < 0.5 \\ -1 & 0.5 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Wavelet Transforms: Wavelet Expansions

- Any function can be represented in terms of scaling and wavelet functions as

$$f(x) = \sum_k c_{j0}(k)\phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_k d_j(k)\psi_{j,k}(x)$$

- If we have orthonormal basis functions then

$$c_{jo}(k) = \int f(x)\phi_{j_0,k}(x)\mathrm{d}x$$

$$d_j(k) = \int f(x)\psi_{j,k}(x)\mathrm{d}x$$

Wavelet Transforms: Discrete Wavelet Transform

- Wavelet Series Expansion becomes the Discrete Wavelet Transform when the input signal is discrete

$$f[x] = \frac{1}{\sqrt{M}} \sum_k W_\phi(j_0, k)\phi_{j_0,k}[x] + \frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k W_\psi(j_0, k)\psi_{j_0,k}[x]$$

- The inverse transforms are given by

$$W_\phi(j_0, k) = \frac{1}{\sqrt{M}} \sum_x f(x)\phi_{j_0,k}(x)$$

$$W_\psi(j_0, k) = \frac{1}{\sqrt{M}} \sum_x f(x)\psi_{j_0,k}(x)$$

- The W_ϕ's are called approximation coefficients and W_ψ's are called the detail coefficients

Wavelet Transforms: Fast Wavelet Transform

- Fast Wavelet Transform exploits the relation between W's at adjacent scaling ($j = j_0$ and $j = j_0 + 1$)
- To derive this relation let us start with the definition

$$W_\phi(j,k) = \frac{1}{\sqrt{M}} \sum_x f(x)\phi_{j,k}(x)$$

- Substitute definition of scaling function

$$W_\phi(j,k) = \frac{1}{\sqrt{M}} \sum_x f(x) 2^{j/2} \phi(2^j x - k)$$

- Using dilation equation and a change of variables $m = 2k + n$

$$W_\phi(j,k) = \frac{1}{\sqrt{M}} \sum_x f(x) \sum_m h_\phi(m - 2k)\sqrt{2}\phi(2^{j+1}x - m)$$

Wavelet Transforms: Fast Wavelet Transform (Cont.)

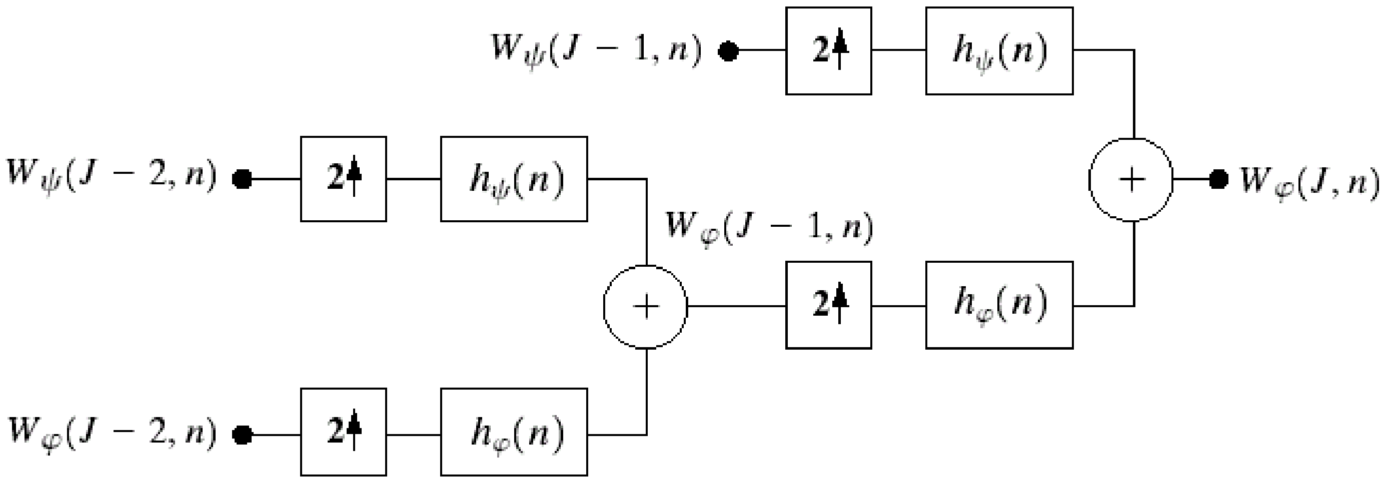
- Rewrite as

$$W_\phi(j,k) = \sum_m h_\phi(m-2k)\frac{1}{\sqrt{M}}\sum_x f(x)\phi(2^{(j+1)/2}x - m)$$

resulting in

$$W_\phi(j,k) = \sum_m h_\phi(m-2k)W_\phi(j+1,m)$$

Wavelet Transforms: Fast Wavelet Transform



Fast Wavelet Transform Implemented with a filter bank

2-D Wavelet Transforms

- Wavelet transform is extended to 2-D easily

$$
\begin{aligned}
f(x,y) &= \frac{1}{\sqrt{MN}} \sum_m \sum_n W_\psi(j_0,m,n)\psi_{j_0,m,n}(x,y) \\
&+ \frac{1}{\sqrt{MN}} \sum_{i=H,V,D} \sum_{j=j_0}^{\infty} \sum_m \sum_n W_\psi^{(i)}(j,m,n)\phi_{j,m,n}^{(i)}(x,y)
\end{aligned}
$$

where

$$\phi_{j,m,n}(x,y) = 2^{j/2}\phi(2^j x - m, 2^j y - n)$$

$$\psi_{j,m,n}^{(i)}(x,y) = 2^{j/2}\psi^{(i)}(2^j x - m, 2^j y - n)$$

with i denoting one of D,H,V components defined as

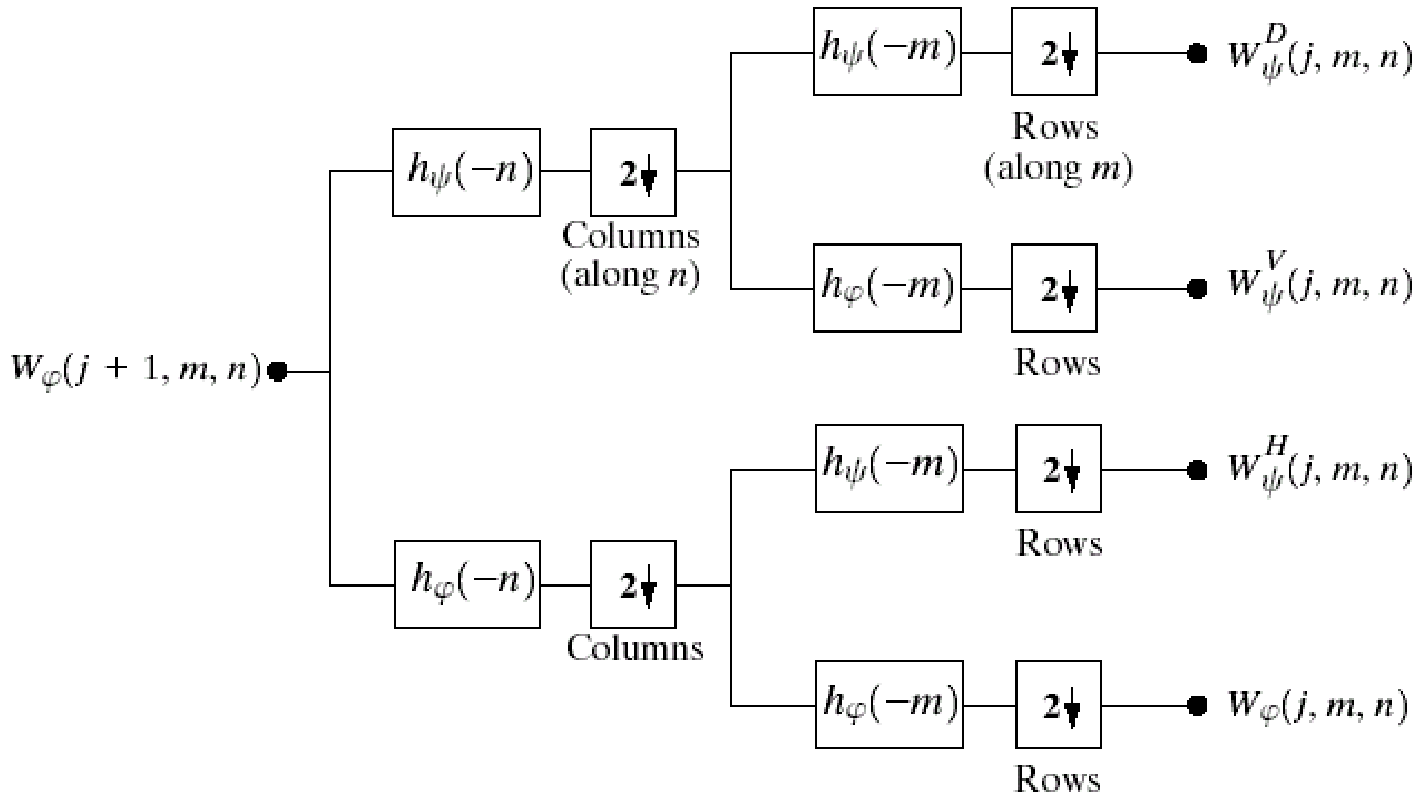
$$\psi^{\mathrm{D}}(x,y) = \psi(x)\psi(y)$$

$$\psi^{\mathrm{H}}(x,y) = \psi(x)\phi(y)$$

$$\psi^{\mathrm{V}}(x,y) = \phi(x)\psi(y)$$

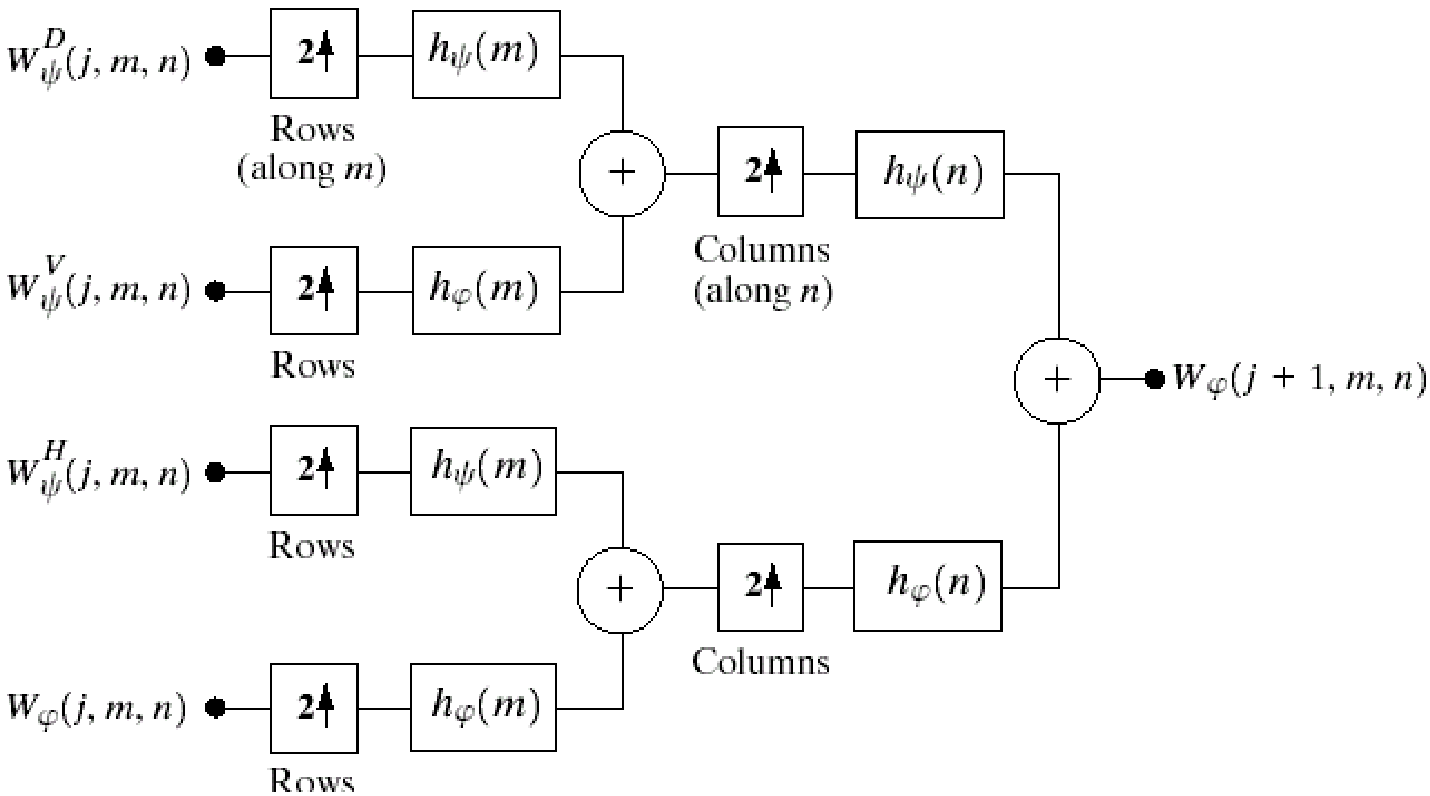
$$\phi^{\mathrm{V}}(x,y) = \phi(x)\phi(y)$$

Wavelet Transforms: 2-D Wavelet Transform



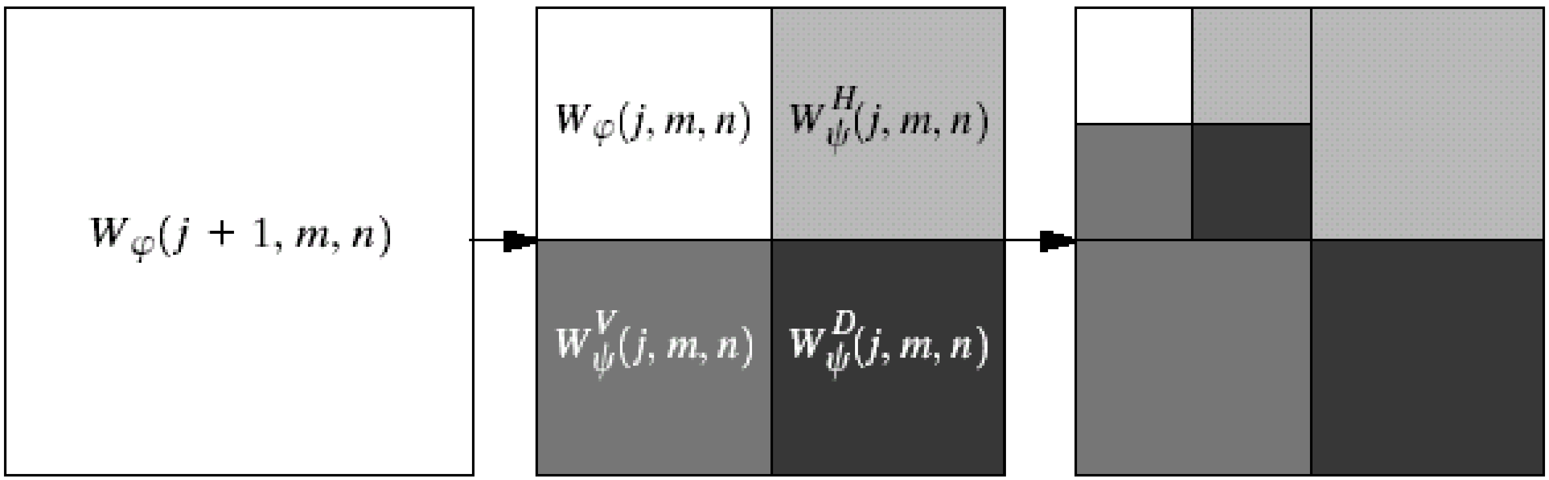
Fast Wavelet Transform Implemented with a filter bank, analysis part

Wavelet Transforms: 2-D Wavelet Transform



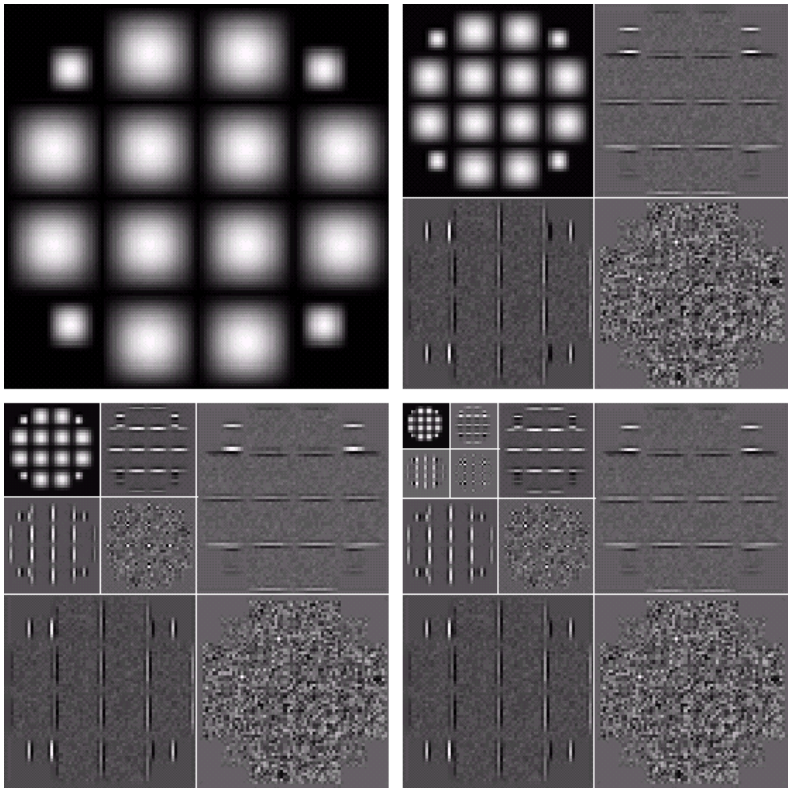
Fast Wavelet Transform Implemented with a filter bank, synthesis part

Wavelet Transforms: 2-D Wavelet Transform



Separation of scaling functions

Wavelet Transform: Image Example



Separation of scaling functions