Outline

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Random Processes/Signals Definition

- A random signal is an extension of a random variable. When a random variable changes value as a function of time, it is called a random signal
- E.g.: Temprature in Chicago on next monday is a random variable
  Temprature in Chicago on mondays is a random signal
- A random process is a collection of ensembles of sample functions (each of these functions are random signals)
- E.g.: Tempratute in different cities on mondays is a random process
- We use x to denote random variables, $x(t)$ to denote random signals, and $X(S,t)$ or $X(t)$ to denote random processes where S is a variable denoting the ensemble (e.g. city name) and t denotes the time variable (e.g. 5th monday)

Moments, Mean, Correlation, and Covariance

- The mth order moment of a random process is defined as

$$\mathrm{E}\{X_{t_i}^p\} = \int x_{t_i}^p p(x_{t_i}) \mathrm{d}x_{t_i}$$

- First few moments have special names:

$$\begin{aligned}
\text{Mean}: \quad & \mathrm{E}\{X_{t_i}\} = m(t_i) = \int x_{t_i} p(x_{t_i}) \mathrm{d}x_{t_i} \\
\text{Autocorrelation}: \quad & \mathrm{E}\{X_{t_i} X_{t_j}\} = R_x(t_i, t_j) \\
& = \int \int x_{t_i} x_{t_j} p(x_{t_i}) p(x_{t_j}) \mathrm{d}x_{t_i} \mathrm{d}x_{t_j} \\
\text{Crosscorrelation}: \quad & \mathrm{E}\{X_{t_i} Y_{t_j}\} = R_{xy}(t_i, t_j) \\
& = \int \int x_{t_i} Y_{t_j} p(x_{t_i}) p(y_{t_j}) \mathrm{d}x_{t_i} \mathrm{d}y_{t_j} \\
\text{Autocovariance}: \quad & \mathrm{E}\{[X_{t_i} - m(t_i)][X_{t_j} - m(t_j)]\} = C_x(t_i, t_j) \\
& = R_x(t_i, t_j) - m(t_i) m(t_j) \\
\text{Crosscovariance}: \quad & \mathrm{E}\{[X_{t_i} - m_x(t_i)][Y_{t_j} - m_y(t_j)]\} = C_{xy}(t_i, t_j) \\
& = R_{xy}(t_i, t_j) - m_x(t_i) m_y(t_j)
\end{aligned}$$

Stationarity

- Stationarity means a steadiness of the characteristics of the random process
- When a random process does not change over time, it is called to be strict sense stationary
- Mathematically, a random process is SSS if

$$p(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = p(x_{t_1+\tau}, x_{t_2+\tau}, \ldots, x_{t_n+\tau})$$

  for all τ

- Strict sense stationarity is a strong condition on a random process

Stationarity (Cont.)

- Weaker stationarity definitions are made: A process is wide sense stationary if first and second order moments do not change in time. Mathematically, a process is WSS if

$$\mathrm{E}\{x_{t_1}\} = \mathrm{E}\{x_{t_1+\tau}\}$$

  and

$$\mathrm{E}\{x_{t_1} x_{t_2}\} = \mathrm{E}\{x_{t_1+\tau'} x_{t_2+\tau'}\}$$

- For $\tau' = -t_2$ the second equation becomes

$$\mathrm{E}\{x_{t_1} x_{t_2}\} = \mathrm{E}\{x_{(t_1-t_2)} x_0\}$$

- Using the short hand notation, we have

$$R_x(t_1, t_2) = R_x(t_1 - t_2) = R_x(\tau)$$

  for a WSS process

Power Density Spectrum

- The frequency characteristics of a stationary random process is analyzed through the power density spectrum defined as:

$$S_x(F) = \int R_x(\tau) e^{-j2\pi F\tau} \mathrm{d}\tau$$

with its inverse

$$R_x(\tau) = \int S_x(F) e^{j2\pi F\tau} \mathrm{d}F$$

- Observe that

$$\mathrm{E}\{X_t^2\} = R_x(0) = \int S_x(F) \mathrm{d}F$$

hence the name

- For WSS random processes the autocorrelation function is even. Therefore, the power density spectrum is always real

Discrete-Time Random Signals

- All developments apply to the discrete case time variable being integers now
- The power density spectrum is again the Fourier transform of the correlation function:

$$S_x(f) = \sum_{m=-\infty}^{\infty} R_x(m) e^{-j2\pi f m}$$

- The autocorrelation is given by the inverse

$$S_x(f) = \int_{-1/2}^{1/2} S_x(f) e^{j2\pi f m} \mathrm{d}f$$

Ergodicity

- Ergodicity means that we can obtain statistical moments from a single realization
- As in stationarity we can have different degrees of ergodicity
- Mean-ergodic processes: mean of the time average is equal to the mean, and variance of the time average is zero for large samples
- The statistical mean is

$$m_x = \mathrm{E}\{X_n\}$$

and the time average is

$$\hat{m}_x = \frac{1}{2N+1} \sum_{n=-N}^{n=N} x(n)$$

- The mean of $\hat{m}_x$ is m_x
- The condition for variance to be zero for large samples is:

$$C_x(m) \to 0 \text{ as } m \to \infty$$

Ergodicity (Cont.)

- Correlation-Ergodic Processes: mean of the time correlation is equal to the statistical correlation, and the variance of the time covariance is zeros for large samples
- Throughout this course, we assume that the random processes we are dealing with are mean-ergodic and correlation-ergodic
- Those two conditions are satisfied with most physical random processes

Innovations Processes

- A method to obtain any WSS process using a linear causal and invertible filter excited by white noise
- Then this linear causal and invertible system will be determined by the WSS process that we want to obtain
- White noise: power density spectrum=σ_w^2
- Linear system: $H(f)$
- Output of the linear system $\sigma_w^2 |H(f)|^2$

Innovations Processes (Cont.)

- We want this output to be the autocorrelation function of any random process:

$$S_x(f) = \sigma_w^2 |H(f)|^2$$

- Switching to z-domain

$$S_x(z) = \sigma_w^2 H(z) H(z^{-1})$$

- Let

$$H(z) = \exp\left\{ \sum_{m=1}^{\infty} v[m] z^{-m} \right\} \quad (1)$$

- We have

$$s_x(z) = \exp\left\{ \sum_{m=-\infty}^{\infty} v[m] z^{-m} \right\}$$

Innovations Processes (Cont.)

- Then, $v[m]$ is the series such that its z transform is $\log[S_x(z)]$
- That is

$$v[m] = \int_{-1/2}^{1/2} \log S_x(f) e^{j2\pi fm} \mathrm{d}f \quad (2)$$

- Given a WSS process, calculate its power density spectrum
- Compute $v[m]$ using Eq. (2)
- The linear system that will result in our random process (with its input being white noise) can be constructed using Eq. (1)