Outline

- Forward Linear Prediction
- Normal Equations
- Solution to Normal Equations
- Backward Linear Prediction
- Optimum Reflection Coefficients

Forward Linear Prediction

- The goal of forward prediction is to obtain a guess of a future value of a random process given its current and past values
- Consider the special case of one-step forward predictor:

$$\hat{x}[n] = -\sum_{k=1}^{p} a_p[k]x[n-k]$$

- The design of a predictor then consists of choosing the coefficients $a_p[i]$ where $i = 1, 2, \ldots, p$
- We can design the our linear predictor by minimizing the error between the predicted and real value

Forward Linear Prediction (cont.)

- This error is

$$f_p[n] = x[n] - \hat{x}[n] = x[n] + \sum_{k=1}^{p} a_p[k]x[n-k]$$

- Combining summation terms and $x[n]$ we obtain

$$f_p[n] = \sum_{k=0}^{p} a_p[k]x[n-k],$$

where we have defined $a_p[0] = 1$

Forward Linear Prediction (Cont.)

- The system that takes $x[n]$ as its input and produces the error function $f_p[n]$ as its output is called the error filter and has the following z-transform

$$A_p(z) = \sum_{k=0}^{p} a_p[k] z^{-k}$$

- The resulting system is

$$F_p(z) = A_p(z) X(z)$$

Direct-form Implementation of Error Filter

- We can implement this error filter using the direct-form

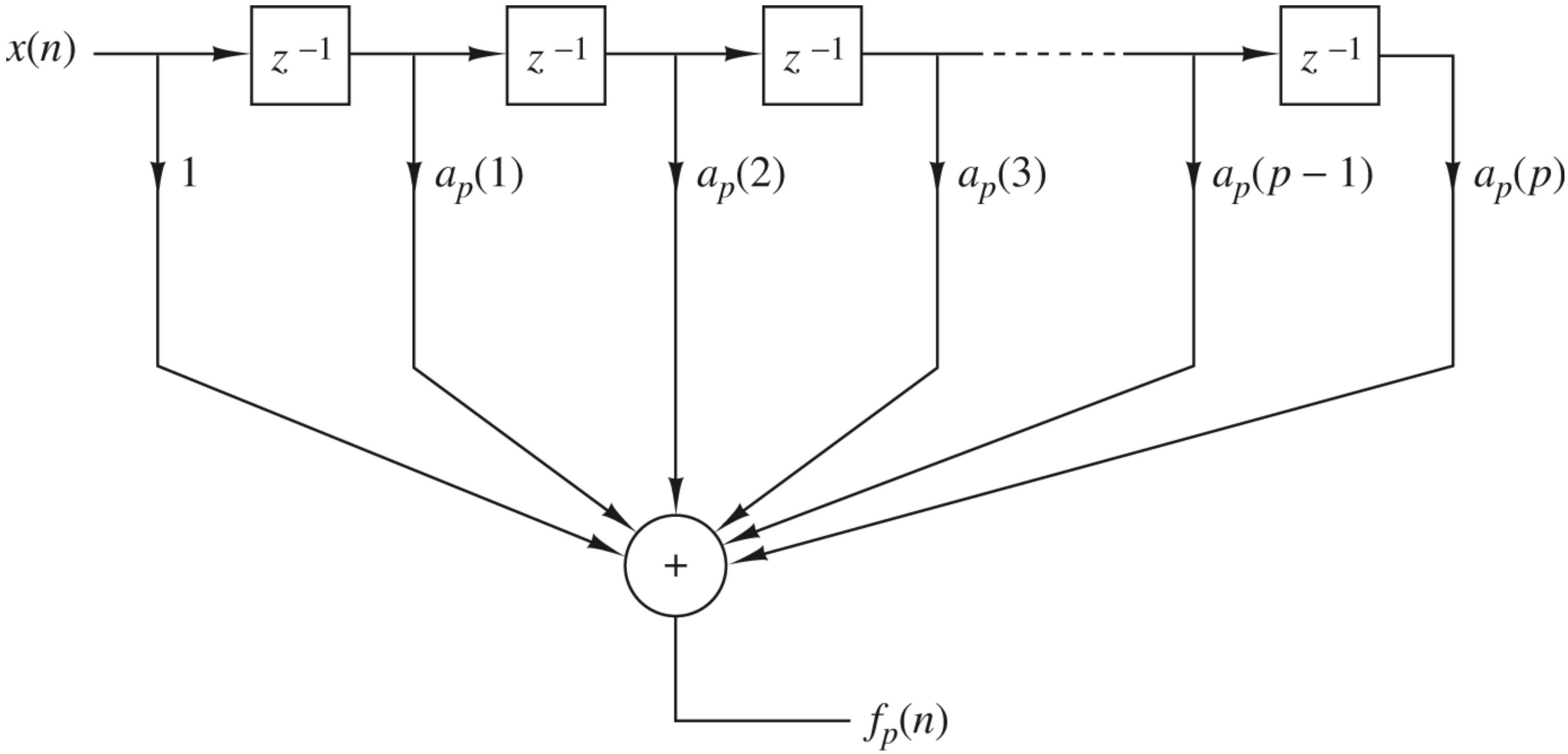

Figure 12.3.2 Prediction-error filter.

Lattice-form Implementation of Error Filter

- And the lattice-form

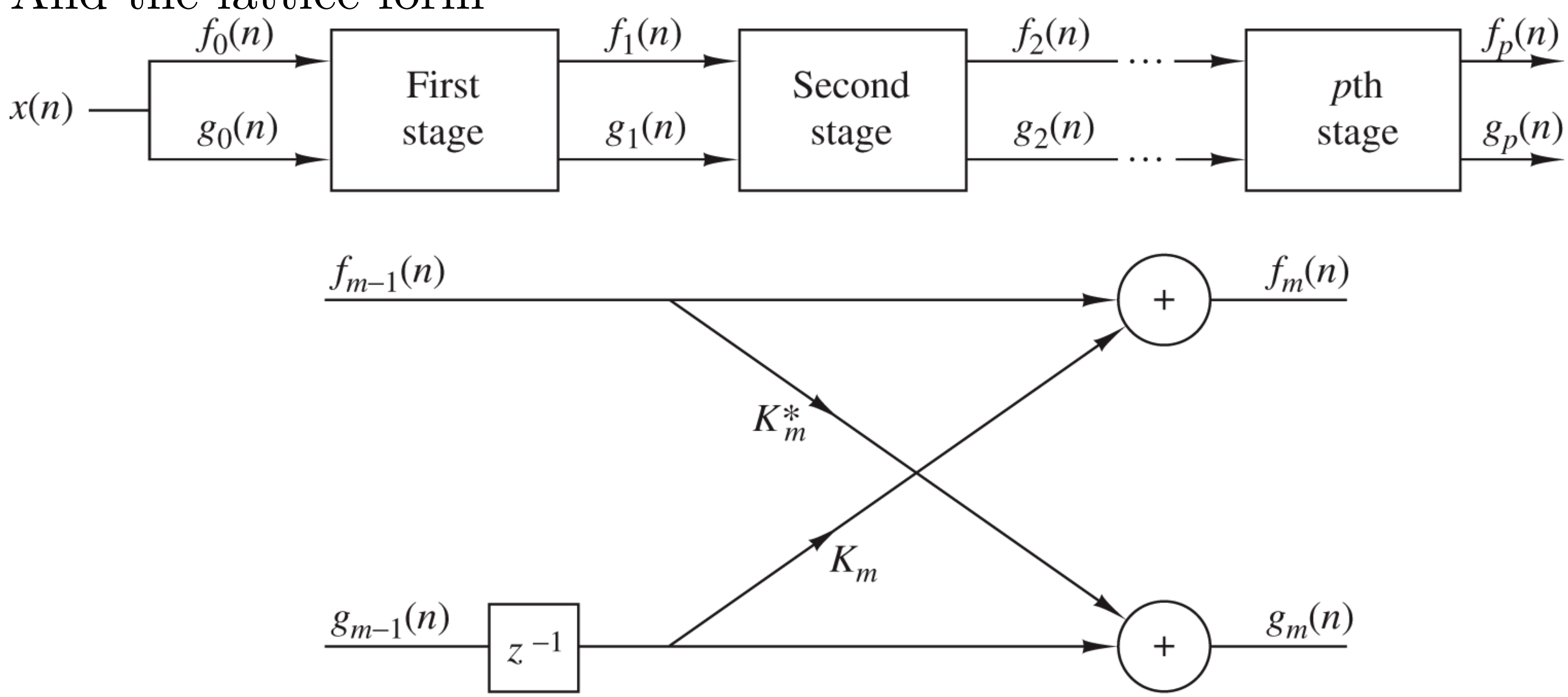

Figure 12.3.3 p-stage lattice filter.

- Lattice form coefficients can be obtained from the direct form coefficients in a recursive fashinon

Normal Equations

- We can design a predictor by minimizing the energy of the error function

$$
\begin{aligned}
a_p &= \arg\min_{a_p} \mathrm{E}\{|f(p)|^2\} \\
&= \arg\min_{a_p} \mathrm{E}\{f_p[n] f_p^*[n]\}
\end{aligned}
$$

- Substituting $f_p[n] = x[n] + \sum_{k=1}^{p} a_p[k] x[n-k]$ we want to minimize

$$
\begin{aligned}
\mathrm{E}\{|x[n]|^2\} &+ 2\mathrm{R}\left\{\sum_{k=1}^{p} a_p^*[k] \mathrm{E}\{x[n] x[n-k]\}\right\} \\
&+ \sum_{k=1}^{p} \sum_{l=1}^{p} a_p^*[l] a_p[k] \mathrm{E}\{x[n-k] x[n-l]\}
\end{aligned}
$$

Normal Equations (Cont.)

- Using the shorthand notation for the correlations, we want to minimize

$$R_x(0) + 2\mathrm{R}\left\{\sum_{k=1}^{p} a_p^*[k] R_x(k)\right\} + \sum_{k=1}^{p}\sum_{l=1}^{p} a_p^*[l] a_p[k] R_x(l-k)$$

Solution to Normal Equations

- The minimization of this function can be obtained by taking the derivative and equating to zero, the resulting equation is

$$R_x(l) = -\sum_{k=1}^{p} a_p[k] R_x(l-k)$$

  called the normal equations

- Substituting this equation into the error function yields the minimum error

$$R_x(0) + \sum_{k=1}^{p} a_p[k] R_x(-k)$$

Backward Linear Prediction

- Now we derive a similar set of equations for the backward linear prediction
- In this case, we want to predict $x[n-p]$ given the later samples $x[n], x[n-1], \ldots, x[n-p+1]$
- We have

$$\hat{x}[n-p] = -\sum_{k=0}^{p-1} b_p[k]x[n-k]$$

Backward Linear Prediction (Cont.)

- The error can be written as

$$g_p[n] = x[n-p] - \hat{x}[n-p] = x[n-p] + \sum_{k=0}^{p-1} a_p[k]x[n-k]$$

- Combining summation terms and $x[n-p]$ we obtain

$$g_p[n] = \sum_{k=0}^{p} b_p[k]x[n-k],$$

  where we have defined $a_p[0] = 1$

- This backward error filter can similarly be realized using a direct or lattice form

Backward Linear Prediction (Cont.)

- The solutions for the backward linear coefficients $b_p[k]$ can similarly be obtained and they are related to the forward linear prediction coefficients:

$$b_p[k] = a_p^*[p-k]$$

- Using this relation we can derive a relation between the Z transforms $G_p(z)$ and $F_p(z)$
- We have

$$\begin{aligned} B_p(z) &= \textstyle\sum_{k=0}^{p} b_p[k] z^{-k} \\ &= a_p^*[p-k] z^{-k} \\ &= z^{-p} A_p^*(z^{-1}) \end{aligned}$$

- These relations allow for a recursive relation between direct-form and lattice-form coefficients

Optimum Reflection Coefficients

- Until now, we considered to minimize the error using a's and b's (direct form coefficients)
- We can minimize the forward prediction error using the lattice structure
- For a single stage, there is only one parameter which is the reflection coefficient K
- For a single lattice

$$f_m[n] = f_{m-1}[n] + K_m g_{m-1}[n-1]$$

- Minimizing the error with respect to K, we find the optimum K for the particular stage

$$K_m = \frac{-\mathrm{E}\{f_{m-1}[n] g^*_{m-1}[n-1]\}}{\mathrm{E}\{|g_{m-1}[n-1]|^2\}}$$

- Observe that the optimum reflection coefficient is the negative of the cross correlation coefficients between forward and backward errors

Practice Problem: 12.2

- Given a random process $x[n]$ with autocorrelation function $R_x[m]$ and the Z-transform of this autocorrelation function

$$\Gamma_x(z) = 9\frac{(z-1/3)(z-3)}{(z-.5)(z-2)},$$

  for $|z|$ between 0.5 and 2

- Determine the filter $H(z)$ that produces this random process from white noise, is this filter unique?

- Determine a stable whitening filter for $x[n]$

Solution to Problem 12.2

- Remember output of a linear system with input being white noise is

$$\sigma_w^2|H(z)|^2 = \sigma_w^2|H(z)||H(z)^{-1}|$$

- Then

$$H(z)H(z^{-1}) = \frac{(z-1/3)(z-3)}{(z-0.5)(z-2)}$$

- We can select

$$H(z) = \frac{z-1/3}{z-0.5}$$

or

$$H(z) = \frac{z-3}{z-2}$$

Solution to Problem 12.2 part (b)

- Whitening filter is the inverse of $H(z) = \frac{1}{H(z)}$
- For a stable whitening filter we must have

$$H(z) = \frac{z - 1/3}{z - 0.5}$$

and the whitening filter is

$$H(z) = \frac{z - 0.5}{z - 1/3}$$