Outline

- Review
- Properties of the Linear-Prediction Error Filters
- Wiener Filters Introduction

Review

- Forward Prediction: finding a guess of $x[n]$ based on $x[n-1], x[n-2], \ldots, x[n-p]$
- The estimate can be written as a linear combination of past values

$$\hat{x}[n] = -\sum_{k=1}^{p} a_p[k]x[n-k]$$

- The optimum solution is the one that minimizes the mean squared error between the true and estimated value. We call this the error function $f_p[n]$

$$f_p[n] = \hat{x}[n] - x[n] = x[n] - \left\{ -\sum_{k=1}^{p} a_p[k]x[n-k] \right\}$$

$$f_p[n] = \sum_{k=0}^{p} a_p[k]x[n-k]$$

Review (Cont.)

- This relation is a linear filtering operation, so we can write everything in the z-domain and implement the filter with any of the block structures that we learned earlier
- However lattice structure has a special property that the new lattice do not change the coefficients of previous lattices

Review (Cont.)

- The optimum linear predictor can be obtained by minimizing the cost function

$$a_{opt}[n] = \arg\min_{a_p[n]} \mathrm{E}\left\{\sum_{n=1}^{p} |f_p[n]|^2\right\} = \arg\min_{a_p[n]} \mathrm{E}\left\{\boldsymbol{f}_p^* \boldsymbol{f}_p\right\}$$

- This minimum can be obtained by taking the derivative and equating to zero resulting in

$$\sum_{k=0}^{p} a_p[k] R_x[l-k] = 0 \qquad l = 1, 2, \ldots, p$$

- These equations are called the Normal Equations

Review (Cont.)

- The normal equations are linear equations that can be solved with any method of matrix inversion
- However it is possible to solve normal equations more efficiently by using the fact that the autocorrelation matrix is Hermitian Conjugate and Toeplitz
- Two such efficient methods are Levinson-Durbin and Schur Algorithm
- Both use some sort of block decomposition of the autocorrelation matrix and result in recursive calculation of the optimum a_p's

Properties of Linear Prediction-Error Filters

- Minimum-phase $\rightarrow$ all zeros inside the unit circle
- Maximum-phase $\rightarrow$ all zeros outside the unit circle
- Remember forward linear predictor-error filter

$$f_p[n] = \sum_{k=0}^{p} a_p[k]x[n-k]$$

with z transform

$$A_p(z) = \sum_{n=1}^{p} a_p[n]z^{-n}$$

- We will show that this filter is minimum phase, that is has all zeros inside the unit circle using induction

Properties of Linear Prediction-Error Filters (Cont.)

- Let us start with 1, we have

$$A_1(z) = 1 + K_1 z^{-1}$$

  and the zero is $-K_1$

- K_1 is a reflection coefficient and less than 1
- Now second step of proof by induction: assume that the zeros are all inside the unit circle for $p-1$. We will show that, then, the zeros are inside the unit circle for p
- The iterative construction of $A(z)$ is

$$A_p(z) = A_{p-1}(z) + K_p z^{-p} A^*_{p-1}(1/z)$$

  for the p step linear prediction

Properties of Linear Prediction-Error Filters (Cont.)

- Let us find the zero's of this function

$$\frac{-1}{K_p} = z^{-p} \frac{A^*_{p-1}(1/z)}{A_{p-1}(z)}$$

- Let us use $A_{p-1}(z) = \Pi_i(z - z_i)$ and

$$\begin{aligned} A^*_{p-1}(1/z) &= \Pi_i[(1/z) - z_i]* \\ &= \Pi_i[(z/|z|^2) - z_i^*] \end{aligned}$$

- The zeros for p are now the solutions to

$$\frac{-1}{K_p} = z^{-p} \frac{\Pi_i[(z/|z|^2) - z_i^*]}{\Pi_i(z - z_i)}$$

- This function has a magnitude larger than 1, when $|z|$ is larger than 1, with $|z_i| < 1$ (coming from previous step
- Since its magnitude is K_p, it cannot be larger than 1, and this concludes the proof by induction

Properties of Linear Prediction-Error Filters (Cont.)

- Now let us have a look at the backward prediction-error filter
- We know that

$$B_p(z) = z^{-p} A_p^*(z^{-1})$$

- Since $A_p(z)$ has zeros all inside unit circle, $B_p(z)$ has zeros all outside unit circle
- Backward linear prediction-error filter is maximum phase

Wiener Filters: Introduction

- Now we are considering the problem of estimating a signal in the presence of noise
- In a lot of cases additive noise model is used

$$x[n] = s[n] + w[n],$$

where $x[n]$ is the measured signal, $s[n]$ is the desired signal, and $w[n]$ is the conteminating noise

- Now, let us filter the measurement $x[n]$

$$y[n] = \mathcal{H}\{x[n]\}$$

Wiener Filters: Introduction (Cont.)

- Depending on the application we want different $y[n]$'s
- In general we have

$$y[n] = s[n + D]$$

- $D = 0$ corresponds to usual filtering, the only goal is to get rid of the noise
- $D > 0$ corresponds to prediction, we are trying to reach a guess of future values