Outline

- Wiener Filters: Introduction
- FIR Wiener Filter
- Orthogonality Principle
- Causal IIR Wiener Filter
- Non-causal IIR Wiener Filter

Wiener Filters: Introduction

- Now we are considering the problem of estimating a signal in the presence of noise
- In a lot of cases additive noise model is used

$$x[n] = s[n] + w[n],$$

  where $x[n]$ is the measured signal, $s[n]$ is the desired signal, and $w[n]$ is the conteminating noise

- Let us filter the measurement $x[n]$

$$y[n] = \mathcal{H}\{x[n]\}$$

Wiener Filters: Introduction (Cont.)

- Depending on the application we want different $y[n]$'s
- In general we have

$$y[n] = s[n + D]$$

- $D = 0$ corresponds to usual filtering, the only goal is to get rid of the noise
- $D > 0$ corresponds to prediction, we are trying to reach a guess of future values

Wiener Filters: Introduction (Cont.)

- We use minimum mean-squared error as the optimality criterion
- Assume that all random signals involved are zero mean and wide sense stationary

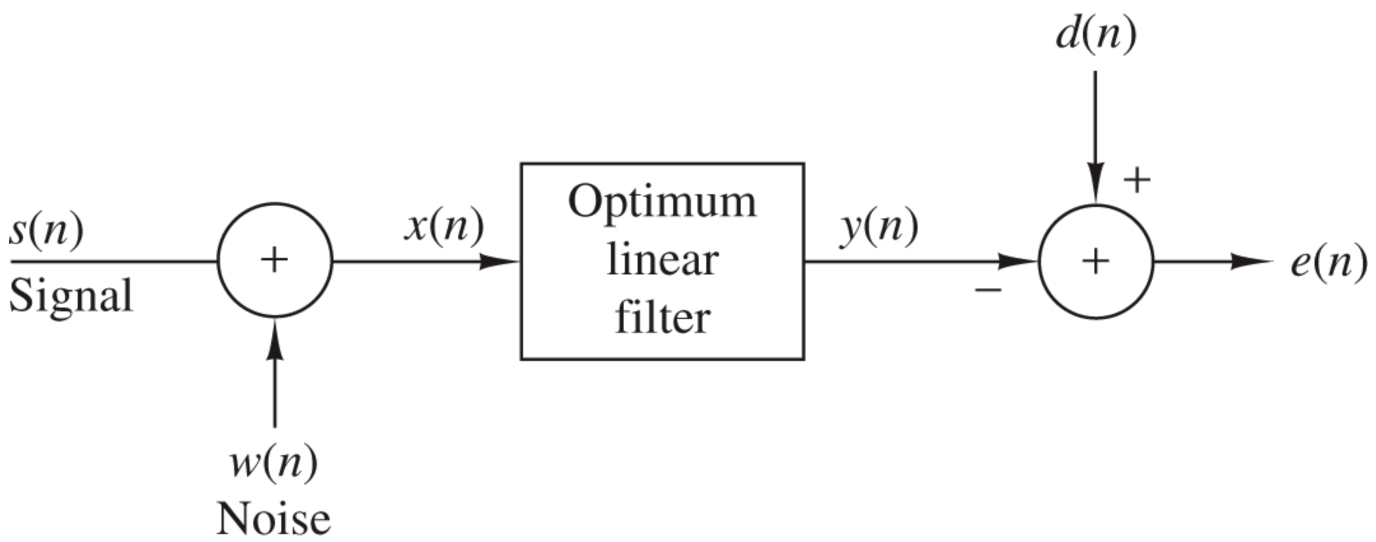

Figure 12.7.1 Model for linear estimation problem.

FIR Wiener Filter

- We restrict the optimum linear filter to be FIR
- Then the filtered signal is

$$y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]$$

- Our criterion: minimum mean-squared error (MMSE)

$$h_{opt} = \arg\min_{h} \mathrm{E}\{|d[n] - \sum_{k=0}^{M-1} h[k]x[n-k]|^2\}$$

- Taking the derivative with respect to h and equating to zero we obtain

$$\sum_{k=0}^{M-1} h[k]R_x[l-k] = R_{dx}[l] \qquad l = 0, 1, \ldots, M-1$$

- A generalized version of normal equations

FIR Wiener Filter (Cont.)

- We can write these equations in matrix form

$$\Gamma_M \boldsymbol{h} = \boldsymbol{r}_d$$

resulting in the solution

$$\boldsymbol{h}_{opt} = \Gamma_M^{-1} \boldsymbol{r}_d$$

- The resulting MMSE can be found by substituting this optimum filter in the error expression

- Consider the expression for the MMSE

$$\begin{aligned}
\epsilon_M &= \mathrm{E}\{|d[n] - \textstyle\sum_{k=0}^{M-1} h[k]x[n-k]|^2\} \\
&= \mathrm{E}\{|d[n]|^2 + \textstyle\sum_{k=0}^{M-1} \sum_{k'=0}^{M-1} h[k]h[k']^* x[n-k]x^*[n-k'] \\
&\quad - \textstyle\sum_{k=0}^{M-1} d[n]h^*[k]x^*[n-k] \\
&\quad - \textstyle\sum_{k=0}^{M-1} d^*[n]h[k]x[n-k]\}
\end{aligned}$$

FIR Wiener Filter (Cont.)

- But we have the following equality for optimum $\boldsymbol{h}$

$$\sum_{k=0}^{M-1} h[k] R_x[l-k] = R_{dx}[l]$$

- Substituting this expression we obtain

$$\mathrm{MMSE}_M = \sigma_d^2 - \sum_{k=0}^{M-1} h_{opt}[k] R_{dx}^*[k]$$

- Substituting $\boldsymbol{h}_{opt} = \Gamma_M^{-1} \boldsymbol{r}_d$ we obtain

$$\mathrm{MMSE}_M = \sigma_d^2 - \boldsymbol{r}_d^{\mathrm{H}} \Gamma_M^{-1} \boldsymbol{r}_d$$

FIR Wiener Filter (Cont.)

- Filtering noise: we have $x[n] = s[n] + w[n]$ and $d[n] = s[n]$
- We can also assume that the signal $s[n]$ and noise $w[n]$ are independent

$$R_x[k] = R_s[k] + R_w[k]$$

$$R_{dx}[k] = R_x[k]$$

- Substituting these normal equations become

$$\sum_{k=0}^{M-1} h[k]\{R_s[l-k] + R_w[l-k]\} = R_s[l]$$

FIR Wiener Filter (Cont.)

- Prediction: we have

$$d[n] = s[n + D]$$

$$R_{dx}[k] = R_s[l + D]$$

- The normal equations become

$$\sum_{k=0}^{M-1} h[k]\{R_s[l-k] + R_w[l-k]\} = R_s[l + D]$$

Orthogonality Principle

- Remember the normal equations

$$\sum_{k=0}^{M-1} h[k]R_x[l-k] = R_{dx}[l] \qquad l = 0, 1, \ldots, M-1$$

- Rearranging terms

$$R_{dx}[l] - \sum_{k=0}^{M-1} h[k]R_x[l-k] = 0$$

- This can be written as

$$\mathrm{E}\{e[n]x^*[n-l]\} = 0$$

where

$$e[n] = d[n] - \sum_{k=0}^{M-1} h[k]x[n-k]$$

$$e[n] = d[n] - \hat{d}[n]$$

Orthogonality Princple

- Orthogonality principle is best explained graphically

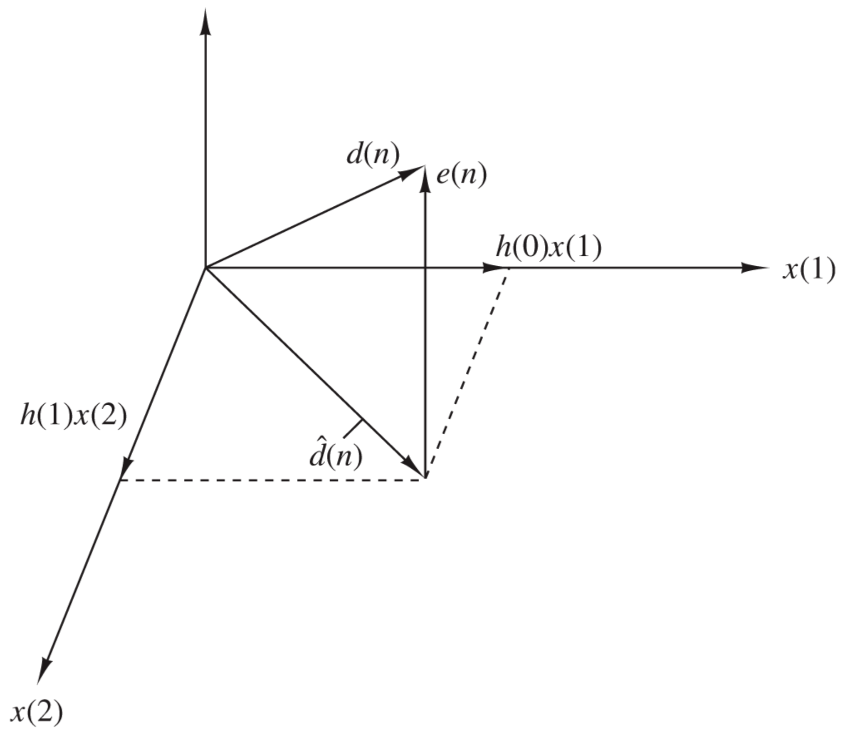

Figure 12.7.2 Geometric interpretation of linear MSE problem.

Causal IIR Wiener Filter

- We now allow the filter to have infinite length

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

- All steps are similar except the summation runs to ∞ leading to normal equations

$$\sum_{k=0}^{\infty} h[k]R_x[l-k] = R_{dx}[l] \qquad l \geq 0$$

- The MMSE for the optimum filter coefficients is

$$\text{MMSE}_M = \sigma_d^2 - \boldsymbol{r}_d^{\text{H}} \Gamma_M^{-1} \boldsymbol{r}_d$$

Causal IIR Wiener Filter (Cont.)

- We cannot use matrix inversion to solve these normal equations since they are of infinite length
- We cannot directly use z-transform neither because the equations are for $l > 0$
- Remember we had this half-infinite summation problem in innovations representation, we will use a similar trick
- The optimum filter h is applied to x, but x can be obtained from white noise with innovations representation, and the inverse of this produces white noise from x

Causal IIR Wiener Filter (Cont.)

- Therefore we can decompose h into two parts: whitening filter and optimum filter

$$x[n] \rightarrow H \rightarrow y[n]$$

$$x[n] \rightarrow \text{Whitening} \rightarrow i[n] \rightarrow Q \rightarrow y[n]$$

- We now have

$$y[n] = \sum_{k=0}^{\infty} q[k] i[n-k]$$

producing

$$\sum_{k=0}^{\infty} q[k] R_i[l-k] = R_{di}[l] \qquad l \leq 0$$

- Using the fact that i is white noise

$$q[l] = \frac{R_{di}[l]}{R_i[0]} = \frac{R_{di}[l]}{\sigma_i^2}$$

Causal IIR Wiener Filter (Cont.)

- To calculate the z-transform of h let us calculate first the z-transform of q and then the z-transform of whitening filter. Product of two will give the z-transform of h

- The z-transform of q

$$\begin{aligned} Q(z) &= \sum_{k=0}^{\infty} q[k] z^{-k} \\ &= \frac{1}{\sigma_i^2} \sum_{k=0}^{\infty} R_{di}[k] z^{-k} \end{aligned}$$

- This is the right sided z-transform of R_{di}

$$[\Gamma_{di}(z)]_+ = \sum_{k=0}^{\infty} R_{di}[k] z^{-k}$$

- We can calculate this as follows: start with the whitening filter

$$i[n] = \sum_{k=0}^{\infty} v[k] x[n-k]$$

Causal IIR Wiener Filter (Cont.)

- We have

$$\begin{aligned} R_{di}[k] &= \mathrm{E}\{d[n]i^*[n-k]\} \\ &= \textstyle\sum_{m=0}^{\infty} v[m]\mathrm{E}\{d[n]x^*[n-m-k]\} \\ &= \textstyle\sum_{m=0}^{\infty} v[m]R_{dx}[k+m] \end{aligned}$$

- Substituting this into definition of z-transform

$$\Gamma_{di}(z) = \sum_{k=-\infty}^{\infty} \{\sum_{m=0}^{\infty} v[m]R_{dx}[k+m]\}z^{-k}$$

- Interchanging order of summations

$$\Gamma_{di}(z) = \sum_{m=0}^{\infty} v[m] \sum_{k=-\infty}^{\infty} R_{dx}[k+m]z^{-k}$$

Causal IIR Wiener Filter (Cont.)

- Change of variable $k' = k + m$

$$\begin{aligned} \Gamma_{di}(z) &= \textstyle\sum_{m=0}^{\infty} v[m] z^{m} \sum_{k'=-\infty}^{\infty} R_{dx}[k'] z^{-k'} \\ &= V(z^{-1}) \Gamma_{dx}(z) \end{aligned}$$

- Finally combining with whitening filter

$$H_{opt}(z) = Q(z) V(z)$$

where

$$Q(z) = \frac{1}{\sigma_i^2} \left[V(z^{-1}) \Gamma_{dx}(z) \right]_{+}$$

Non-causal IIR Wiener Filter

- We now allow the filter to multiply future values as well

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- All steps are similar except the summation runs to ∞ leading to normal equations

$$\sum_{k=-\infty}^{\infty} h[k]R_x[l-k] = R_{dx}[l] \qquad l \epsilon Z$$

- We can use z-transform directly in this case

$$H(z) = \frac{\Gamma_{dx}(z)}{\Gamma_{xx}(z)}$$