Outline

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Performance Analysis of Non-parametric Power Spectrum Estimation

- Let us evaulate three methods that we discussed
- Performance measure: variance, we use the normalized variance with the signal power

$$V = \frac{\mathrm{var}[P_x(f)]}{[\mathrm{E}[P_x(f)]]^2}$$

- Periodogram: the variance is

$$S_x^2(f)\left[1+\left(\frac{\sin 2\pi f N}{N sin 2\pi f}\right)^2\right]$$

and the mean is

$$\int_{-0.5}^{0.5} S_x(\theta) W_B(f-\theta)\mathrm{d}\theta$$

- The ratio goes to 1 as N increases to infinity

Performance Analysis: Bartlett Method

- The variance of the Bartlett estimate is

$$\frac{1}{K} S_x^2(f) \left[1 + \left(\frac{\sin 2\pi f M}{M sin 2\pi f} \right)^2 \right]$$

- The mean is

$$\int_{-0.5}^{0.5} S_x(\theta) W_B(f-\theta) \mathrm{d}\theta$$

where

$$W_B(f) = \frac{1}{M} \left(\frac{\sin \pi f M}{\sin \pi f} \right)^2$$

- If we calculate the value of the performance measure as $N, M \rightarrow \infty$ we find that it is $1/K$
- Can be made very small with large K
- However large K results in a very poor frequency resolution: tradeoff

Performance Analysis: Welch Method

- Variance of two examples of Welch estimate

$$\begin{cases} \frac{1}{L} S_x^2(f), & \text{no overlap} \\ \frac{9}{8L} S_x^2(f), & \text{50\% overlap, triangular window} \end{cases}$$

- The mean is

$$\int_{-0.5}^{0.5} S_x(\theta) W_B(f-\theta) \mathrm{d}\theta$$

where

$$W(f) = \frac{1}{MU} \left| \sum_{n=0}^{M-1} w[n] e^{-j2\pi f n} \right|^2$$

- Calculating the performance measure as $N, M \rightarrow \infty$, we have

$$\left\{ \begin{array}{ll} \frac{M}{N}, & nooverlap \\ \frac{9M}{16N}, & 50\% overlap \end{array} \right\}$$

- Tradeoff between frequency resolution and estimation performance

Performance Analysis: Blackman-Tuckey

- We have approximate expressions for the variance

$$S_x^2(f)\left[\frac{1}{N}\sum_{m=-(M-1)}^{M-1} w^2[m]\right]$$

  and the mean

$$\int_{-0.5}^{0.5} S_x(\theta)W(f-\theta)\mathrm{d}\theta$$

  where W is the windowing function

- An example of triangular window results in the performance criterion value $\frac{2M}{3N}$
- All methods exhibit a tradeoff characteristics between frequency resolution and the estimation performance
- If both are considered, Bartlett performs poorly compared to other two methods

Parametric Power Spectrum Estimation: Parametric Modeling

- Remember we related the PSD to autocorrelation using a finite summation
- This is equivalent to assuming that the autocorrelatin values are zero outside the summation range, which is not true in reality
- This, along with windowing, was required since we estimated the autocorrelation sample by sample
- We can avoid all these by using a parametric model for the random signal

Parametric Power Spectrum Estimation: Parametric Modeling (Cont.)

- We will model the random signal as being the output of an ARMA system in general

$$H(z) = \frac{\sum_{k=0}^{q} b_k^{-k}}{1 + \sum_{k=1}^{p} a_k z^{-k}}$$

resulting in the difference equation

$$x[n] = -\sum_{k=1}^{p} a_k x[n-k] + \sum_{k=0}^{q} b_k w[n-k]$$

where $w[n]$ is assumed to be white noise

- Then, the PSD is

$$S_x(f) = |H(f)|^2 S_w(f) = \sigma_w^2 |H(f)|^2 = \sigma_w^2 \frac{|B(f)|^2}{A(f)|^2}$$

- With this modeling, we have reduced the power spectrum estimation to the estimation of a_k's and b_k's

Relation Between Model Parameters and Autocorrelation

- Remember

$$x[n] = -\sum_{k=1}^{p} a_k x[n-k] + \sum_{k=0}^{q} b_k w[n-k]$$

- Then the autocorrelation is

$$R_x[m] = \begin{cases} -\sum_{k=1}^{p} a_k R_x[m-k], & m > q \\ -\sum_{k=1}^{p} a_k R_x[m-k] + \sigma_w^2 \sum_{k=0}^{q-m} h[k] b_{k+m}, & m \leq 0 \leq q \\ R_x^*[-m], & m < 0 \end{cases}$$

  for a general ARMA model

- These equations can be written in matrix form and can be solved efficiently

Yule-Walker Algorithm: AR Model

- Consider the AR version of the model

$$R_x[m] = \begin{cases} -\sum_{k=1}^{p} a_k R_x[m-k], & m > 0 \\ -\sum_{k=1}^{p} a_k R_x[m-k] + \sigma_w^2, & m = 0 \\ R_x^*[-m], & m < 0 \end{cases}$$

- Then we have the following equation

$$\begin{bmatrix} R_x(0) & R_x(-1) & \ldots & R_x(1-p) \\ R_x(1) & R_x(0) & \ldots & R_x(2-p) \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ R_x(p-1) & R_x(p-2) & \ldots & R_x(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_p \end{bmatrix} = - \begin{bmatrix} R_x(1) \\ R_x(2) \\ . \\ . \\ . \\ R_x(p) \end{bmatrix}$$

- The model parameters can be determined using fast matrix inversion algorithms such as Levinson-Durbin

Selection of AR Model Order

- We model the the random process using an AR model, but how complex should this AR model be?
- A too simple AR model would overly smooth (simplify) the spectrum
- A too complex AR model would introduce unwanted peaks meaning that we have orders that we do not need
- One method is to look at the error decrease as a function of the order, and select the order when the decrease in error is no longer significant

Selection of AR Model Order (Cont.)

- We need to use a criterion which balances the high number of parameters with a penalty
- One such criterion is Akaike's information criterion

$$\mathrm{AIC}(p) = \log(sig\hat{m}a_{wp}^2) + 2p/N$$

- Increasing model order p decreases first term but increases second term, giving an optimum p value
- There are several other model selection criterion such as minum description length, cirterion autoregressive transfer, etc

MA Model

- For a moving average model, the relation between the autocorrelation and the model parameters are

$$R_x[m] = \begin{cases} 0, & m > q \\ \sigma_w^2 \sum_{k=0}^{q} b_k b_{k+m} + \sigma_w^2, & 0 \leq m \leq q \\ R_x^*[-m], & m < 0 \end{cases}$$

- Since for MA model, autocorrelation is zero for $m > q$, we can use the approach in non-parametric power spectrum estimation and use

$$P_x(f) = \sum_{m=-q}^{q} R_x[m] e^{-j2\pi f m}$$

MA Model (Cont.)

- Alternatively, the MA model can be approximated with a high order AR model

$$B(z) = \frac{1}{A(z)}$$

- Then we have $A(z)B(z) = 1$ resulting in $a * b = \delta[0]$:

$$\sum_{k=0}^{q} b_k \hat{a}_{n-k} = \delta[0]$$

- Order for MA model can similary be selected based on some model fitness criterion

ARMA Model

- Remember the most general relation between autocorrelation and ARMA model parameters

$$R_x[m] = \begin{cases} -\sum_{k=1}^{p} a_k R_x[m-k], & m > q \\ -\sum_{k=1}^{p} a_k R_x[m-k] + \sigma_w^2 \sum_{k=0}^{q-m} h[k] b_{k+m}, & m \le 0 \le q \\ R_x^*[-m], & m < 0 \end{cases}$$

- We can solve for AR part of model parameters using the values $m > q$
- For improved performance we will solve the overdetermined system of equations, assuming we have realiable estimates based upto lag M

$$\begin{bmatrix} R_x[q] & R_x[q-1] & \dots & R_x[q-p+1] \\ R_x[q+1] & R_x[q] & \dots & R_x[q-p+2] \\ . & . & \dots & . \\ R_x[M-1] & R_x[M-2] & \dots & R_x[M-p] \end{bmatrix}$$

ARMA Model (Cont.)

- We know that cascading ARMA model with AR model produces MA model
- Hence, let us filter x (output of ARMA model) with AR filter to produce

$$v[n] = x[n] + \sum_{k=1}^{p} \hat{a}_k x[n-k]$$

- This signal $v[n]$ is output of an MA model, so we can apply MA power spectrum estimation methods
- Last step is to obtain P_x by

$$P_x(f) = \frac{P_v(f)}{\hat{A}^2(f)}$$