Outline

Review:

Signals and Systems

Probability Theory

A Bit of Estimation

Signals and Representations of Them

- A signal f can be represented using several basis functions:

$$f(u) = \int K(u, u') f(u') \mathrm{d}u'$$

- Common basis functions:
  - $K(u, u') = \delta(u, u') \rightarrow$ time/space domain
  - $K(u, u') = \exp(-juu') \rightarrow$ frequency domain
    where the variable u is now called the frequency
- Obtaining (calculating) one representation from the other one is performed by signal transformation

Analogy to Language

- Think of the signal as a meaning of the word (regardless of a specific language). Then, different basis functions correspond to different languages, and signal transforms (and their inverses) correspond to language translations
- Just as a language may be more suitable to tell a particular story, different transforms may be more suitable for different applications
- Contrast: There is no perfect language translation (subjective), but signal transforms are perfect (meaning that well-defined, one to one, objective)

Systems

- A system is any kind of mathematical or physical operation that changes an input signal f to produce the output signal g
- We show g=H{f} in an abstract way
- Linearity: A system is called linear when

$$a_1 g_1 + a_2 g_2 = a_1 H\{f_1\} + a_2 H\{f_2\}$$

  for $g_1 = H\{f_1\}$ and $g_2 = H\{f_2\}$
- Causality: A system is called causal when its output depends on only the present and past values of the input

Linear Systems

- We will deal with linear systems in this class most of the time if not always
- Mathematically g can be calculated from f and H in different ways depending on the representation we choose
- Example, space/time domain:

$$g(u) = \int f(u')H(u, u')\mathrm{d}u'$$

- Watch for the similarity between signal transforms. Mathematically, a signal transform and a linear system is equivalent
- In reality, in a signal transform, signal is not changed only its representation; whereas for the linear system case the signal itself is altered

Time/Space Invariance

- A system is called time/space invariant when a shift in the input signal causes the same shift in the output signal
- In the space domain:

$$g(u - u_0) = \int f(u' - u_0) H(u, u') \mathrm{d}u'$$

given $g(u) = \int f(u') H(u, u') \mathrm{d}u'$ (*)

- Start with (*)

$$\begin{aligned} g(u - u_0) &= \int f(u') H(u - u_0, u') \mathrm{d}u' \\ &= \int f(u' - u_0) H(u, u') \mathrm{d}u' \end{aligned}$$

- Change of variables $u'' = u' - u_0$

$$\int f(u') H(u - u_0, u') \mathrm{d}u' = \int f(u'') H(u, u'' + u_0) \mathrm{d}u''$$

- That is $H(u - u_0, u') = H(u, u' + u_0)$ The linear system depends only on the difference between the two variables, so we can write
- $g(u) = \int f(u')H(u - u')\mathrm{d}u'$
- This equation is called the convolution, the output of a linear system is obtained by convolving the linear system function with the input only when it is time/space invariant

Time/Space Invariant Systems in the Frequency Domain

- Remember

$$g(u) = \int f(u')h(u-u')\mathrm{d}u'$$

- Let us take the FT of both sides

$$\int g(u)\exp(-juv)\mathrm{d}u = \int\int f(u')h(u-u')\mathrm{d}u'\exp(-juv)\mathrm{du}$$

-

$$\begin{aligned} G(v) \quad = \quad & \int f(u')h(u-u') \\ & \exp[-j(u-u')v]\exp[j(u-u')v]\exp[-juv]\mathrm{d}u'\mathrm{d}u \end{aligned}$$

- Now we have

$$G(v) = \int f(u') \exp[-ju'v] \int h(u-u') \exp[-j(u-u')v] \mathrm{d}u \mathrm{d}u'$$

Finally

$$\begin{aligned} G(v) &= \int f(u') \exp(-ju'v) H(v) \mathrm{d}u' \\ &= F(v)H(v) \end{aligned}$$

Discrete Time Systems

- Discrete time systems can either be obtained by sampling a continuous time signal, or they arise in intermediate steps of discrete time signal processes: $f[n] = f(Tn)$
- $f[n]$ is called the discrete time signal obtained by sampling $f(u)$ at sampling points Tn with n being an integer, and T is called the sampling period
- We will review sampling theorem in detail before starting multirate signal processing

Discrete Fourier Transform

- The discrete Fourier Transform of a signal is given by:

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp\{-j2\pi kn/N\}$$

with the inverse

$$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k] \exp\{j2\pi kn/N\}$$

Discrete Fourier Transform

- Similar to the continuous case, we have a discrete time convolution for shift invariant systems:

$$g[n] = \sum_{n'=0}^{N} h[n-n']f[n']$$

- This corresponds to a multiplication in the frequency domain

$$G[k] = H[k]f[k]$$

Continuous Fourier Series

- Fourier Series is a way to represent continuous periodic signals

$$f(t) = \sum_k c_k e^{ikt}$$

- c_k's are called Fourier series coefficients which can be calculated

$$c_k = \frac{1}{2\pi} \int f(t) e^{-ikt} dt$$

- This periodic signal can be made aperiodic by extending the period to infinity. This after certain manipulations and definitions give us the regular continuous FT

$$F(f) = \int f(t) e^{-i2\pi tf} dt$$

with inverse FT

$$f(t) = \int F(f) e^{i2\pi tf} df$$

Discrete Fourier Series

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$$F(w) = \int f[n] e^{-iwn} dt$$

with inverse FT

$$f[n] = \frac{1}{2\pi} \int_{period} F(w) e^{iwn} dw$$

- DFT is formed by the samples of DTFT

Summary

- Discrete Fourier Series and Continuous Fourier Series always correspond to discrete signals (coefficients), for discrete and continuous periodic signals
- When this period is infinity at the limit, the FT and DTFT are formed form aperiodic continous and discrete time signals. FT is aperiodic, DTFT is periodic.
- Periodicity in one domain results in discreteness in other domain and vise versa
- When DTFT is sampled DFT is obtained. DFT itself is periodic and discrete, a transform for limited discrete signals.

Probability Theory

- A random variable x is a variable that takes different values for different realizations (unlike deterministic variables where their values are fixed)
- A random process x_n is a collection of random variables indexed with one or more indices
- A probability density function $P(x_0)$ is simply the probability that the random process can take a particular value:

$$P(x_0) = \text{Probability that } x_n = x_0$$

Properties of PDF

- $$P(x_0 \epsilon S) = \int_S P(x_n) \mathrm{d}x_n$$

- $$\int P(x_n) \mathrm{d}x_n = 1$$

- Cumulative distribution function $C(x_n)$ is defined as
$$C(x_n) = \int_{-\infty}^{x_n} P(x'_n) \mathrm{d}x'_n$$

- CDF is the integral of PDF, therefore PDF is the derivative of the CDF owing to the fundamental law of calculus

- Most of the time, calculating CDF's are much easier. So, we calculate CDF's and take the derivative to obtain PDF's: Derived PDF method

Independence, Stationarity, Ergodicity

- Two random processes are said to be independent if and only if

$$P(x_n, x_m) = P(x_n).P(x_m)$$

- A random process is said to be stationary if the PDF of a function does not change over time:

$$P(x_{n_1}, x_{n_2}, \ldots, x_{n_N}) = P(x_{n_1+n_0}, x_{n_2+n_0}, \ldots, x_{n_N+n+0})$$

- A process is called ergodic when the time averages and the mean are equivalent:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} x_k = \int x_n P(x_n) \mathrm{d}x_n$$

Conditional Probability and Bayes' Rule

- Conditional probability: $P(A/B) = P(A)$ given B
- $P(A/B) = P(A,B)/P(B)$. If A and B are independent $P(A/B) = P(A,B)/P(B) = P(A)P(B)/P(B) = P(A)$
- Bayes' Rule: a relation between prior and posterior probabilities

$$P(A/B) = \frac{P(B/A)P(A)}{P(B)}$$

Statistical Moments

- We often use statistical moments instead of the full PDF to understand and analyze a random process
- Mean: $\mu_{x_n} = \int x_n P(x_n) \mathrm{d}x_n$
- Variance: $\int (x_n - \mu_{x_n}) P(x_n) \mathrm{d}x_n$
- Mean and variance completely determines some PDF's such as the Normal(Gaussian) PDF, but this is not true in general
- Autocorrelation $\phi_{xx} : E\{x_{n+m} x_n^*\}$
- Autocovariance: $E\{(x_{n+m} - \mu_x)(x_n - \mu_x)^*\}$
- Crosscorrelation $\phi_{xy} : E\{x_{n+m} y_n^*\}$
- Autocorrelation: $E\{(x_{n+m} - \mu_x)(y_n^* - \mu_y)^*\}$

Properties of Autocorrelation Function

- Max at zero
- Monotonically decreasing
- When a random process x passes through a linear system with frequency response $H(v)$ producing another random process y we have the relation $\phi_{yy} = |H(v)|^2\phi_{xx}$

Estimation

- In probability theory, we do not talk about the origin of variables involved such as mean, variance, or other functions of PDF's
- When it comes to real life we need to find out the values of such quantities
- This is called estimation
- Generally and roughly speaking, we have access to some observations which are random
- Based on these observations, estimation is the process of calculating a guess of what e.g. an input signal, a parameter of the input signal, or some parameters of a system are

Detection

- Detection is a special case of estimation, the only parameter we want to estimate is the existence or the non-existence of a signal
- Example: we listen to a phone line and try to decide weather there is a speech signal present or not
- Example: we look at a water supply and try to find out if it contains harmful chemicals

Different Philosophies of Estimation

- Stochastic Estimation: we assume that the quantity we want to estimate is stochastic, hence we estimate its moments, or pdf
- Deterministic Estimation: We assume that the quantity we want to estimate is deterministic but unknown
- Bayesian Estimation: We assume prior knowledge about the signal to be estimated, such as the PDF of the signal
- Non-bayesian Estimation: We assume that we do not know anything about the signal to be estimated
- Bayesian estimation of course has the advantage of utilizing prior knowledge, but this may sometimes bias the estimation towards an incorrect estimate (depending on the accuracy of our prior knowledge)

Basic Steps of Estimation

- Construct a mathematical model, including the parametrization of the quantity/signal of interest
- Construct a cost function, such as the negative likelihood or mean squared error
- Calculate or develop a method of minimizing this cost function