Outline

- Discrete time systems (FIR and IIR) and difference equations
- Z-transform
- Recursive and non-recursive realizations
- FIR structures
  - Direct-form
  - Cascade-form
  - Lattices

Discrete Time Systems

- The relation between the input and output of a discrete time system can be written as a convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Causality: output depends on present or past values of input:

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

- For a causal FIR system, we have finite number of coefficients:

$$y[n] = \sum_{k=n-(M-1)}^{n} x[k]h[n-k]$$

Recursive vs Non-recursive Systems

- Non-recursive system: Output depends only on the input
- Recursive system: Output depends on past values of the output as well
- Example: summation $\rightarrow$ a linear system

$$
\begin{aligned}
y[n] &= \textstyle\sum_{k=0}^{n} x[k] \\
y[n+1] &= \textstyle\sum_{k=0}^{n+1} x[k] = \sum_{k=0}^{n} x[k] + x[n+1] \\
y[n+1] &= y[n] + x[n+1] \\
y[n] &= y[n-1] + x[n]
\end{aligned}
$$

- That is you can change between recursive (last line) and non-recursive (first line) implementations of the same system

Z Transform

- Definition:

$$X[z] = \sum_{-\infty}^{\infty} x[n] z^{-n}$$

- The DFT is a special case of the Z-transform. Calculating Z transform at $z = e^{\frac{j2\pi k}{N}}$ results in the DFT.
- The Z transform of $h[n]$ is called the system function
- Example:

$$y[n] = 0.5y[n-1] + x[n]$$

$$Y[z] = 0.5Y(z)z^{-1} + X[z]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.5z^{-1}}$$

- All these (difference equation, convolution sum, impulse response, system function) are different valid representation of the same linear system.

Inverse Z-transform

- Although Z-transform has a mathematical inverse transformation formula, we usually use look up tables
- Z-Transforms are converted into fractions, and each fraction's z-transform is looked up from a table
- Works since most practical Z-transforms are rational functions

FIR Implementations: Direct Form

- $$y[n] = \sum_{k=0}^{M-1} b_k x[n-k]$$
- $$H(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$

FIR Implementations: Direct Form

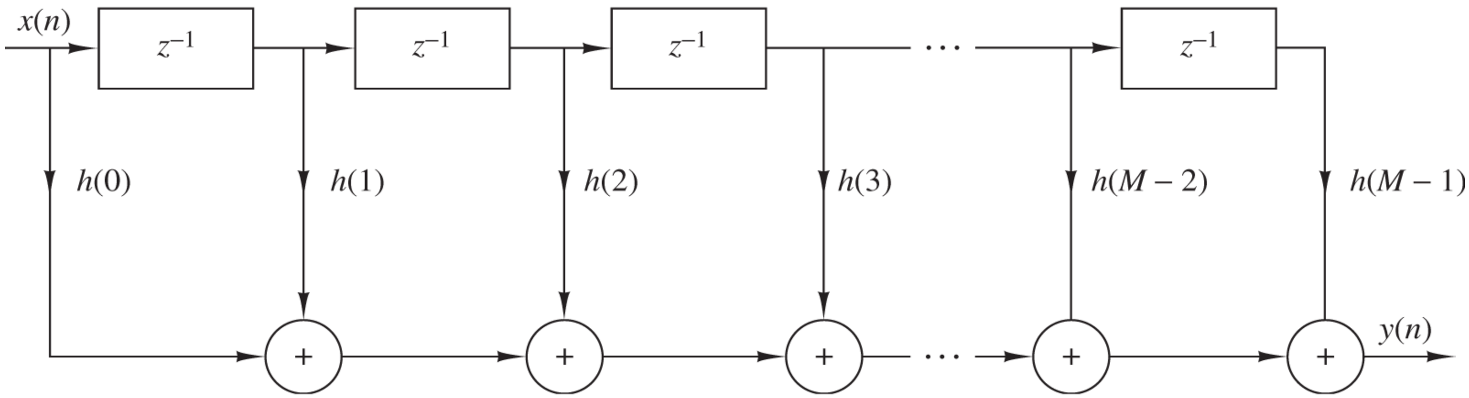


Figure 9.2.1 Direct-form realization of FIR system.

FIR Implementations: Cascade Form

- $H(z)$ can be written as a multiplication of smaller systems:

$$H(z) = \Pi_{k=1}^{N} H_k(z) = \Pi_{k=1}^{N}(b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2})$$

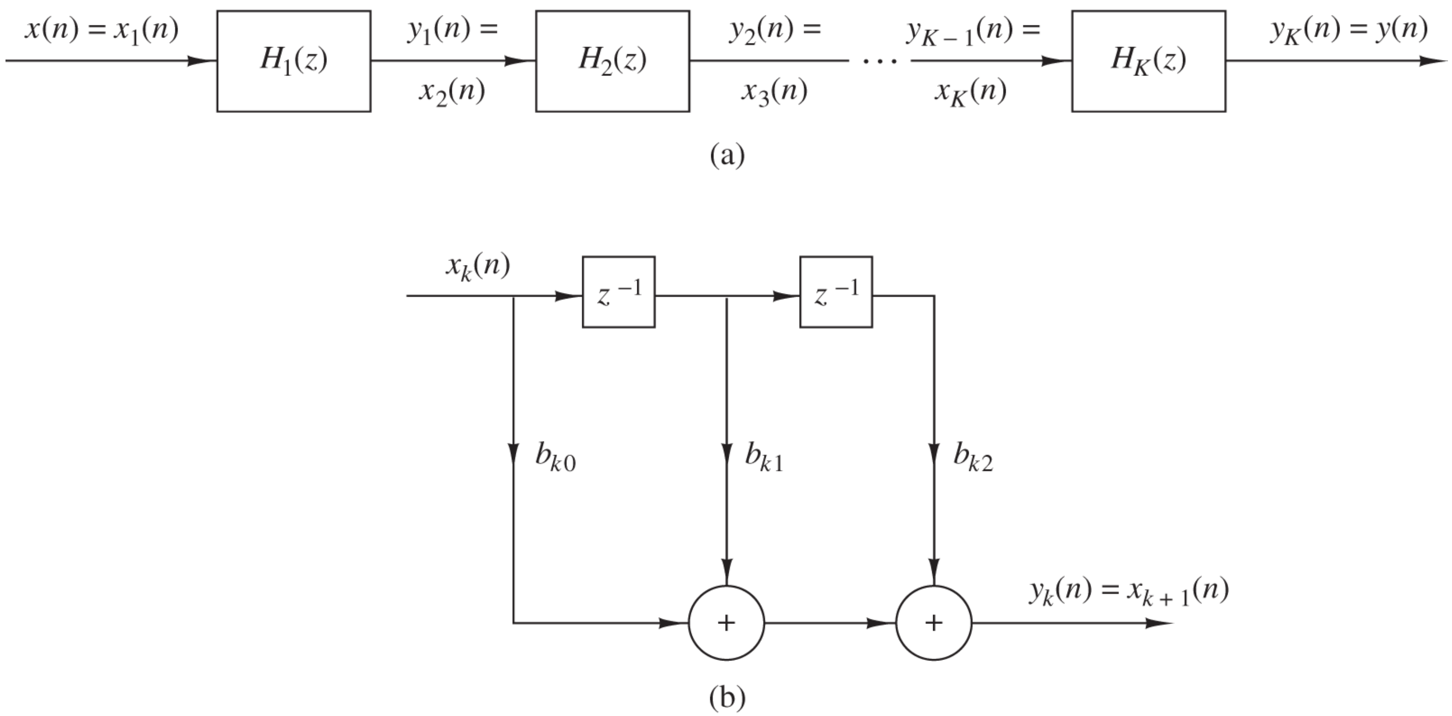
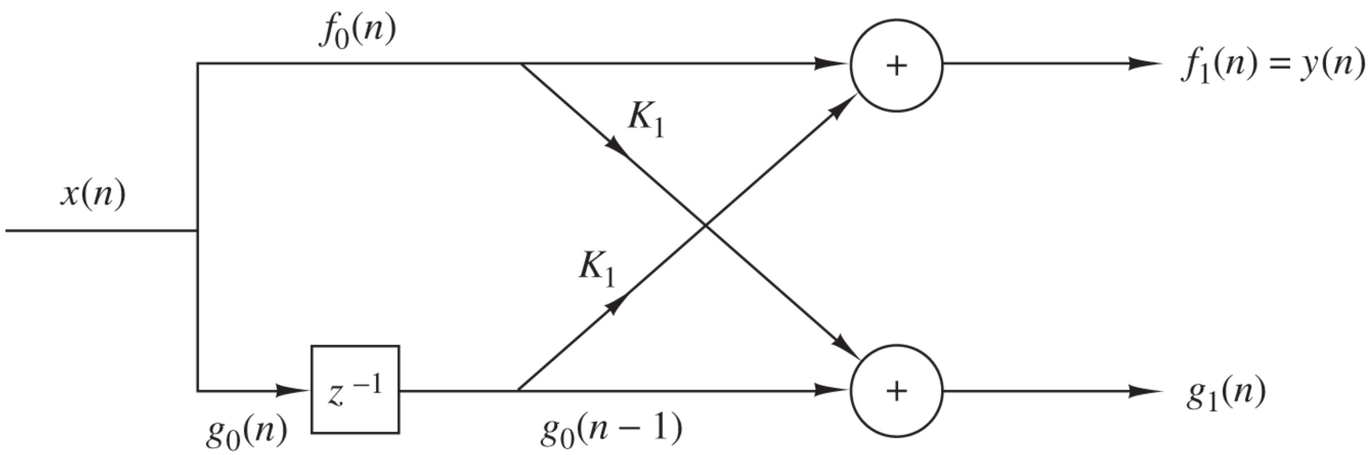


Figure 9.2.3 Cascade realization of an FIR system.

FIR Implementations: Lattices

- We can implement FIR filters in a different way again using smaller structures called lattices.



$$f_0(n) = g_0(n) = x(n)$$
$$f_1(n) = f_0(n) + K_1 g_0(n-1) = x(n) + K_1 x(n-1)$$
$$g_1(n) = K_1 f_0(n) + g_0(n-1) = K_1 x(n) + x(n-1)$$

Figure 9.2.9 Single-stage lattice filter.

FIR Implementations: Lattices

- When we consider an arbitrary lattice

$$
\begin{aligned}
f_m[n] &= f_{m-1}[n] + K_m g_{m-1}[n-1] \\
g_m[n] &= K_m f_{m-1}[n] + g_{m-1}[n-1]
\end{aligned}
$$

FIR Implementations: Lattices

- Assuming that the relation between g and x is defined by a system $A(z)$, and the relation between g and x is defined by a system $B(z)$; there is a relation between $A(z)$ and $B(z)$:

$$B_m(z) = z^{-m} A_m(z^{-1})$$

- This means coefficients are reversed for $f_m[n]$ and $g_m[n]$
- This can be seen by calculating $g_2[n]$ and then recursively calculating $g_m[n]$, and making the above observation

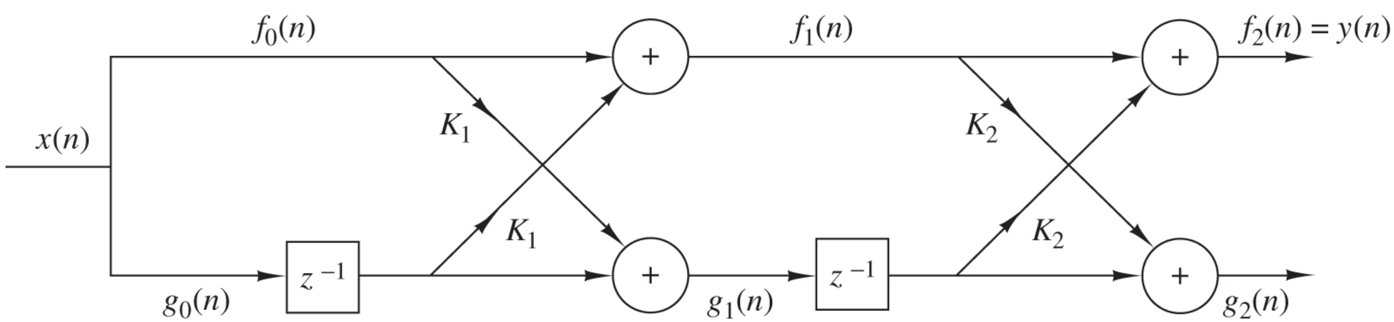


Figure 9.2.10 Two-stage lattice filter.

FIR Implementations: Lattices, Lattice form to Direct form

- The recursive relation between the lattice coefficients and direct form coefficients are then:

$$\begin{aligned}
A_0(z) &= B_0(z) = 1 \\
A_m(z) &= A_{m-1}(z) + K_m z^{-1} B_{m-1}(z) \\
B_m(z) &= z^{-m} A_m(z^{-1}) \\
B_m(z) &= K_m A_{m-1}(z) + Z^{-1} B_{m-1}(z)
\end{aligned}$$

- Direct form coefficients are the coefficients of the final $A_m(z)$ that is obtained

FIR Implementations: Lattices, Direct form to Lattice form

- The recursive relation between direct form coefficients and the lattice form coefficients are:

$$\begin{aligned} B_m(z) &= z^{-m} A_m(z^{-1}) \\ A_{m-1}(z) &= \tfrac{A_m(z) - K_m B_m(z)}{1 - K_m^2} \end{aligned}$$

- These are obtained by simply calculating the inverse of the recursive relations between the lattice form coefficients and the direct form coefficients

- At each iteration K_m's are the coefficients of the largest power of $A_m(z)$

FIR Implementations: Lattices

- All smaller lattices get combined to form the final system again using smaller structures called lattices.

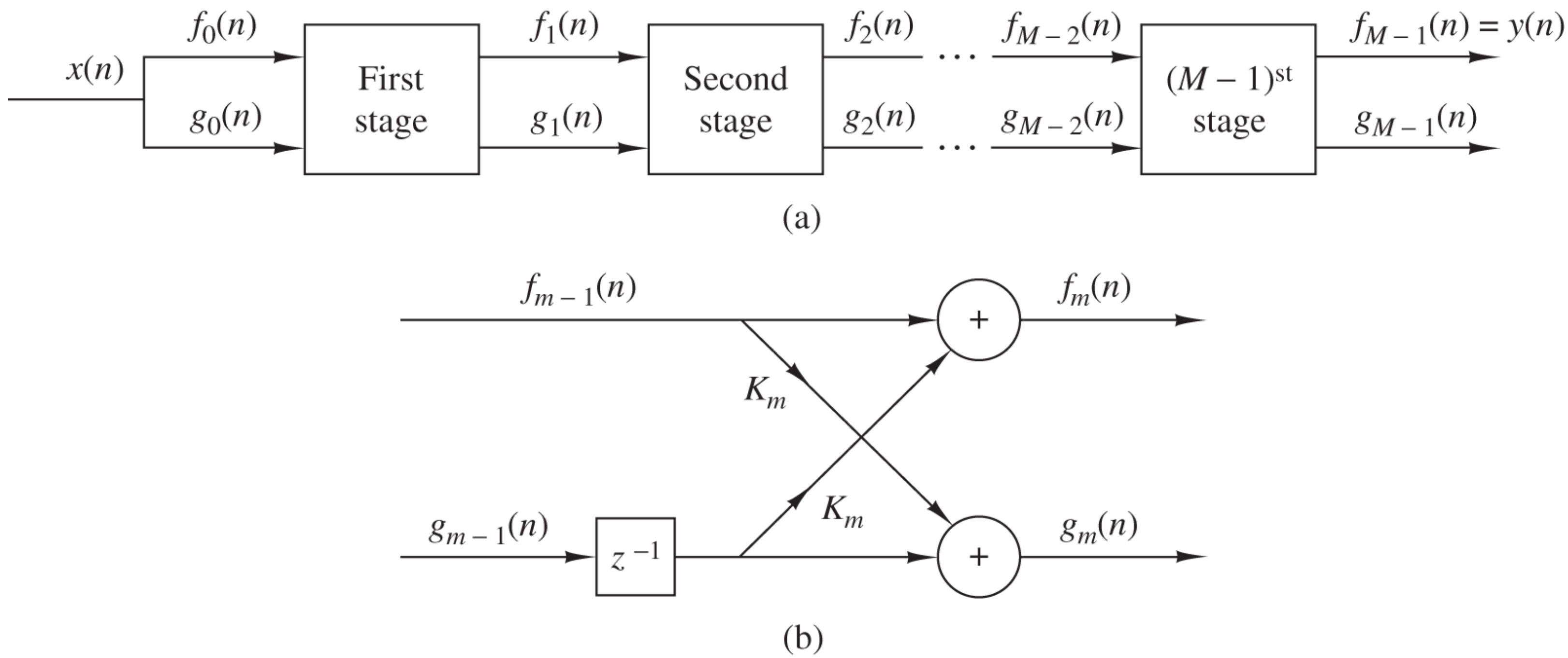


Figure 9.2.11 $(M-1)$-stage lattice filter.