Outline

- Optimum FIR filters
- Comparison of FIR filter design methods
- IIR filter design based on continuous filters
  - Approximation of derivatives
  - Impulse variance
  - Bilinear transformation

Optimum FIR Filters

- The goal is to design the filter such that it is optimum in a certain desired property
- As an example we can choose the filter that results in "the minimum of the maximum error"
- We create cost function (error) by weighting the difference betweend the desired filter response and the FIR filter we create
- Mathematically error $E(w)$ is

$$E(w) = W(w)[H_d(w) - H(w)]$$

Optimum FIR Filters

- Then optimum filter coefficients can be found by

$$\alpha_{\text{opt}} = \arg\min_{\alpha_k}[\max|E(w)|]$$

- This optimization problem can be solved with alternation theorem given that our FIR filter can be written as

$$H(w) = Q(w)\sum_{k=0}^{L}\beta_k\cos(wk)$$

  where β_k is linearly related to filter coefficients α_ks, and $Q(w)$ is one of $1, \cos(w/2), \sin(w), \sin(w/2)$ depending on the filter being symmetric/antisymmetric, and having even/odd number of coefficients

Optimum FIR Filters - Example of $P(w)$ and $Q(w)$

- Assume we have a symmetric filter with even number of coefficients $h[n] = h[M-1-n]$
- Then DFT can be written as

$$H(w) = 2 \sum_{n=0}^{M/2-1} h[n] \cos[w(\frac{M-1}{2} - n)]$$

- To simplify define $b[k] = 2h[M/2-k]$ and obtain

$$H(w) = \sum_{k=1}^{M/2} b[k] \cos(wk - w/2)$$

Optimum FIR Filters - Example of $P(w)$ and $Q(w)$

- Finally, using trigonometric idendities

$$H(w) = \cos(w/2) \sum_{k=0}^{M/2-1} \alpha_k \cos(wk)$$

- For this example $Q(w) = cos(w/2)$ which can be blended into $W(w)$ to leave us with the error function

$$E(w) = \tilde{W}(w)[\tilde{H}_d(w) - P(w)]$$

Optimum FIR Filters - Alternation Theorem

- The min-max problem is guaranteed to have a solution if there exists enough number of frequencies where we have extermums,
- This is possible when the error function oscillates between minimum and maximum errors allowed
- Then our optimization problem can be posed as a linear problem

$$\tilde{W}(w_n)[\tilde{H}_d(w_n) - P(w_n)] = (-1)^n \delta$$

where δ is our maximum allowed error

- Rearranging terms

$$P(w_n) + \frac{(-1)^n \delta}{\tilde{W}(w_n)} = H_d(w_n)$$

- This is a linear system of equations with respect to filter coefficients α_k which are inside $P(w)$ for known w_n

Optimum FIR Filters - Alternation Theorem

- However, we do not know w_ns at the begining so we
  - initialize by guessing w_ns
  - solve the system, find coefficients and $P(w)$
  - find w_ns by using this $P(w)$
  - repeat until convergence

Comparison of FIR Filter Design Methods

- Window design intitutiave but not good control over critical frequencies
- Frequency sampling solve this problem by giving us control over critical frequencies
- However, frequency sampling do not have control over what happens in between the samples, that is the ripples
- Optimum FIR filter design with alternation theorem allows us to both control critical frequencies and ripples

IIR Filter Design Introduction

- Huge filter design methods to create continuous filters with Laplace transform

$$H_a(s) = \frac{B(s)}{A(s)}$$

  which represents a filter that is characterized by constant coefficient differential equations

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

- The Laplace transforms $B(s)$ and $A(s)$ are based on α_ks and β_ks as $B(s) = \sum_{k=0}^{M} \beta_k s^k$ and $A(s) = \sum_{k=0}^{M} \alpha_k s^k$
- We would like to use this body of methods to create digital filters
- All we need is a way to transfer continuous filters to digital filters
- There are multiple ways to do this “disceretization” or sampling

IIR Filter Design by Impulse Invariance

- Here we simply obtain the digital filter by uniformly sampling the continuous filter

$$h[n] = h_a(nT)$$

- Using sampling theorem, the frequency response of the digital filter is the repeated version of the analog filter

$$H(f) = F_s \sum_{k=\infty}^{\infty} H_a[(f-k)F_s]$$

- What is the mapping between s and z which we can use to substitute in $H_a(s)$ to obtain $H(z)$
- FT : $s = j\omega = jw/T$, DFT $z = e^{jw}$, then we have mapping $z = e^{sT}$ only for FT not general z and s
- Left hand plane $\rightarrow$ inside unit circle

IIR Filter Design by Impulse Invariance

- There is an elegant relation between poles in continuous and discrete domains with impulse variance
- Consider

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$$

  with time domain representation $h_a(t) = \sum_{k=1}^{N} c_k e^{p_k t}$

- Sampling this we obtain digital filter as

$$h[n] = \sum_{k=1}^{N} c_k e^{p_k n T}$$

IIR Filter Design by Impulse Invariance

- Taking the z transform

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

  and this has poles $e^{p_k T}$

- Not a simple relation for zeros, remember $z = e^{sT}$ is not a direct mapping that can be used to relate analog and continuous filters

IIR Filter Design by Approximation of Derivatives

- Instead of directly sampling the time domain filter, we approximate the differential equations by using digital approximations to derivatives

$$y'(t) = \frac{y(nT) - y(nT - T)}{T} = \frac{y[n] - y[n-1]}{T}$$

- Taking the laplace and z-transform of both sides

$$s = \frac{1 - z^{-1}}{T}$$

- This can be generalized to

$$s^k = \left(\frac{1 - z^{-1}}{T}\right)^k$$

- Then mapping $s = \frac{1-z^{-1}}{T}$ can be used to create a digital filter $H(z)$ from an analog filter $H_a(s)$
- Stability preserved (left hand plane corresponds to a region inside unit circlt)

IIR Filter Design by Bilinear Transformation (Approximation of Integrals)

- Consider the system

$$H(s) = \frac{b}{s+a}$$

- This corresponds to the system

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

- Using an integral to obtain $y(t)$ we have

$$y(t) = \int_{t_0}^{t} y'(\tau)d\tau + y(t_0)$$

- For a single sampling interval we have

$$y[nT] = \frac{T}{2}[y'(nT) + y'(nT-T)] + y[nT-T]$$

- We know what $y'[nT]$ is from the given differential equation

$$y'[nT] = -ay[nT] + bx[nT]$$

IIR Filter Design by Bilinear Transformation (Approximation of Integrals)

- Substituting we have

$$\left(1+\frac{aT}{2}\right)y[n]-\left(1-\frac{aT}{2}\right)y[n-1]=\frac{bT}{2}[x[n]-x[n-1]]$$

- Taking the $z-transform$ we obtain

$$H(z)=\frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}$$

- Then, the mapping for bilinear transformation is

$$s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$