Outline

- Multirate Signal Processing Introduction
- Decimation by an integer
- Interpolation by an integer

Multirate Signal Processing Introduction

- A digital signal sampled from an analog signal, or as part of a digital signal processing system needs to be resampled sometimes as part of a processing method, and sometimes directly for producing output
- This resampling can be done directly in the digital domain, or by conversion to analog world first and then resampling back to digital domain
- Although multirate signal processing can be performed purely in digital domain, it is easier to understand when the underlying continuous signal is considered

Multirate Signal Processing Introduction

- Digital to analog conversion is possible by the perfect interpolation formula

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(nT)g(t - nT)$$

- Perfect reconstruction is possible if sampling rate is higher than Nyquist rate (remember not required)

- Once this continuous signal is obtained, resampling can easily be performed as

$$x_r[m] = x_a(mT_r) = \sum_{n=-\infty}^{\infty} x[n]g(mT_r - nT)$$

where subscript "r" denotes resampling

- This resulting resampling formula is a purely digital operation performed in discrete world

Multirate Signal Processing Introduction

- Let us rewrite the sampling rate conversion formula

$$x_r[m] = \sum_{n=-\infty}^{\infty} x[n] g\left(T\left(\frac{mT_r}{T} - n\right)\right)$$

- Separating the $\frac{mT_r}{T}$ into integer k_m and fractional Δ_m parts

$$x_r[m] = \sum_{n=-\infty}^{\infty} x[n] g((k_m + \Delta_m - n)T)$$

- Changing the summation index $k = k_m - n$

$$x_r[m] = \sum_{k=-\infty}^{\infty} g(kT + \Delta_m T) x[k_m - k]$$

- This can be seen as a convolution, input is $x[n]$ output is $x_r[n]$, the system is $g_m[n] = g(nT + \Delta_m t)$
- Watch! this is a linear system, but time-varying, g depends on m (in addition to n)

Multirate Signal Processing Introduction

- We have

$$x_r[m] = \sum_{k=-\infty}^{\infty} g(kT + \Delta_m T)x[k_m - k]$$

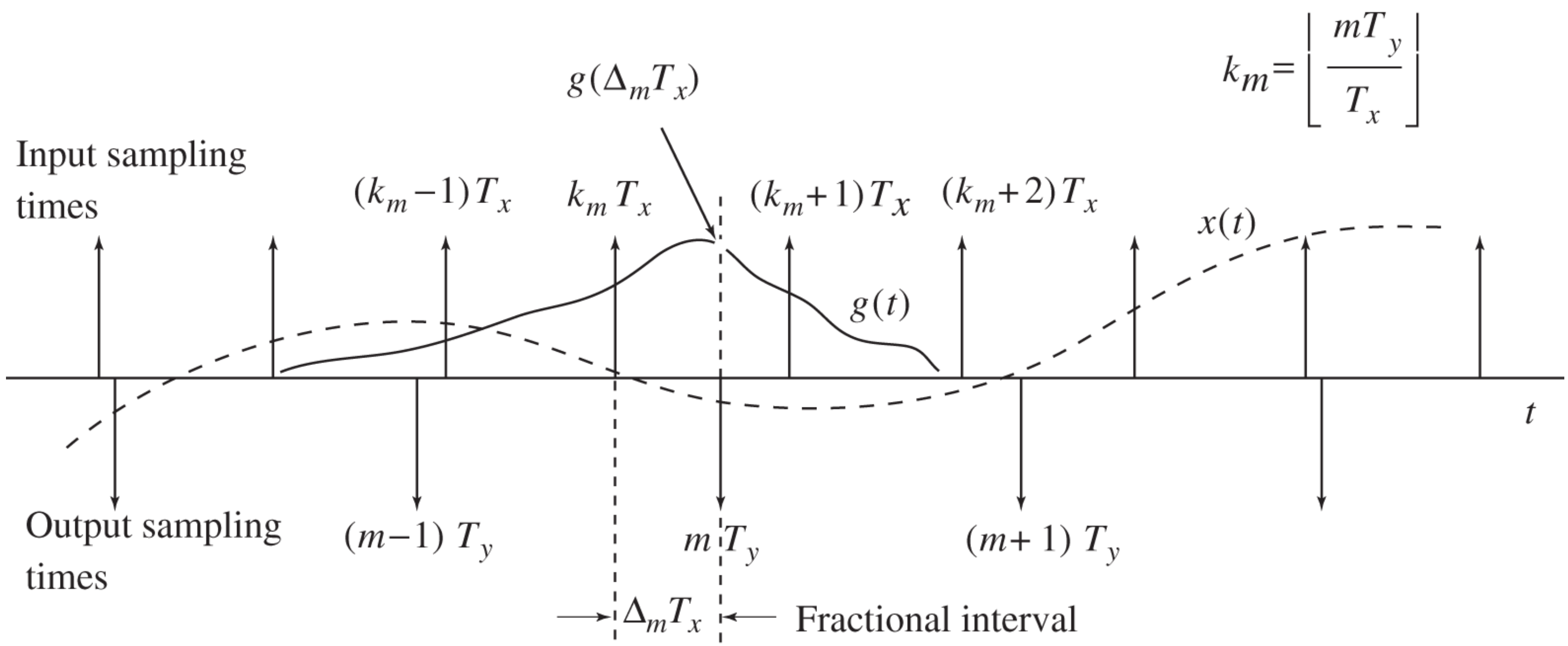

Figure 11.1.1 Illustration of timing relations for sampling rate conversion.

Multirate Signal Processing Introduction

- The system is

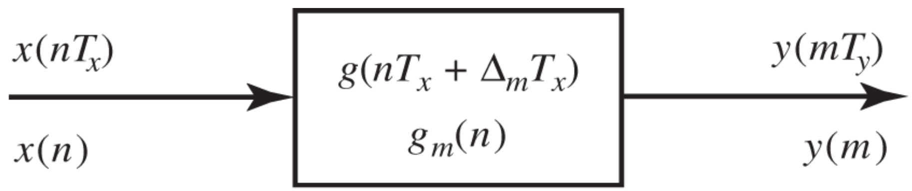

Figure 11.1.2 Discrete-time linear time-varying system for sampling rate conversion.

Decimation by an Integer Factor "D"

- If we directly downsample the signal (pretty much like sampling the discrete signal), we will have aliasing
- To avoid this aliasing we need to do a prefiltering with

$$H_D(w) = \left\{ \begin{array}{ll} 1 & |w| < \pi/D \\ 0 & \text{otherwise} \end{array} \right\}$$

- In this way the signal that is inputted to the downsampler is bandlimited to π/D and downsampling by D does not cause aliasing

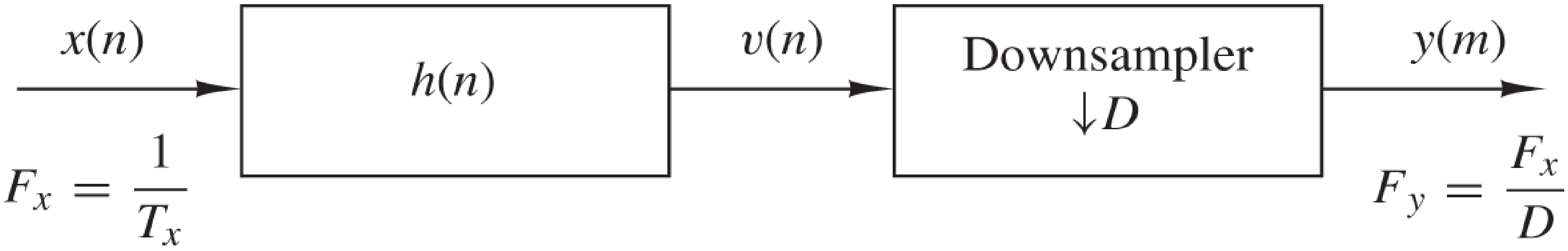

Figure 11.2.1 Decimation by a factor D.

Decimation by an Integer Factor "D"

- The output of this decimator is

$$x_d[m] = v[mD] = \sum_{k=0}^{\infty} h[k]x[mD-k]$$

- This is again a time-varying sytem in general when considered to be a relation between $x_r[n]$ and $x[n]$
- This is the relation between $x[m]$ and $x_d[m]$ in time domain. How about the relation between the two in the z-domain?
- We have

$$X_d(z) = \sum_{m=-\infty}^{\infty} x_d[m]z^{-m}$$

$$X_d(z) = \sum_{m=-\infty}^{\infty} \tilde{v}[mD]z^{-m}$$

where $\tilde{v}$ is a signal that is a repetetion of $v[m]$ every D sample and zero otherwise

Decimation by an Integer Factor "D"

- By making a change of variable $m' = mD$

$$X_d(z) = \sum_{m'=-\infty}^{\infty} \tilde{v}[m'] z^{-m'/D}$$

- This repetetition (forming of $\tilde{v}$ can be seen as v multiplying by a train of impulses

$$X_d(z) = \sum_{m=-\infty}^{\infty} v[m] \left[\frac{1}{D} \sum_{k=0}^{D-1} e^{j2\pi mk/D} \right] z^{-m'/D}$$

which is

$$X_d(z) = \frac{1}{D} \sum_{k=0}^{D-1} \sum_{m=-\infty}^{\infty} v[m] e^{j2\pi mk/D} z^{-m'/D}$$

Decimation by an Integer Factor “D”

- This can be seen as a z-transform of v as

$$X_d(z) = \frac{1}{D} \sum_{k=0}^{D-1} V(e^{-j2\pi k/D} z^{1/D})$$

or

$$X_d(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} z^{1/D}) H_D(e^{-j2\pi k/D} z^{1/D})$$

Decimation by an Integer Factor "D"

- This z-transform can be converted to FT relation by evaluating on the unit circle

$$X_d(w) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{w - 2\pi k}{D}\right) H_D\left(\frac{w - 2\pi k}{D}\right)$$

- If H_D is chosen such that there is no aliasing

$$X_d(w) = \frac{1}{D} X\left(\frac{w}{D}\right)$$

Decimation by an Integer Factor "D"

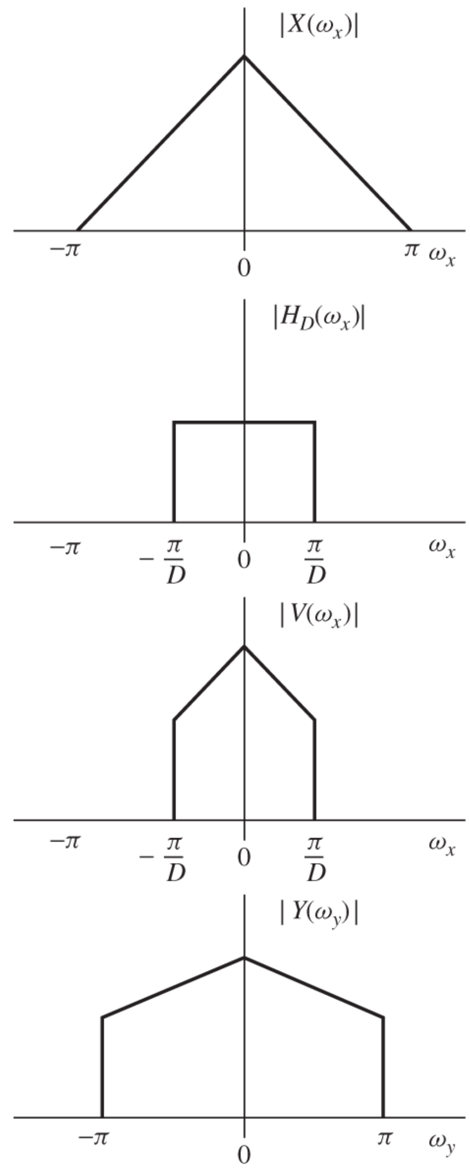

Figure 11.2.3 Spectra of signals in the decimation of $x(n)$ by a factor D.

Interpolation by an Integer Factor "I"

- Now, we perform a series of similar derivations to see the relation between an original signal and its interpolated version in time and frequency domains
- Let us choose our interpolator to be the one that preserves the spectral shape of the signal
- An intermediate signal $v[n]$ that represents zero filled version of our original signal

$$v[m] = \left\{ \begin{array}{ll} x[m/l] & m = 0, l, 2l, \ldots \\ 0 & \text{otherwise} \end{array} \right\}$$

Interpolation by an Integer Factor "I"

- Then the z-transform can be written as

$$V(z) = \sum_{m=infty}^{\infty} v[m] z^{-m}$$

which is

$$\sum_{m=infty}^{\infty} x[m] z^{-mI} = X(z^I)$$

- In the frequency domain

$$V(w) = X(wI)$$

Interpolation by an Integer Factor "I"

- To preserve the shape of x and get rid of the spillover terms that comes from wI, we use a lowpass filter

$$H_I(w) = \left\{ \begin{array}{ll} C & 0 \leq |w| \leq \pi/I \\ 0 & \text{otherwise} \end{array} \right\}$$

- In the time domain we have

$$x_I[m] = \sum_{k=-\infty}^{\infty} h[m-k]v[k]$$

- Considering that v is zero in between every I sample

$$x_I[m] = \sum_{k=-\infty}^{\infty} h[m-kI]x[k]$$

Interpolation by an Integer Factor "I"

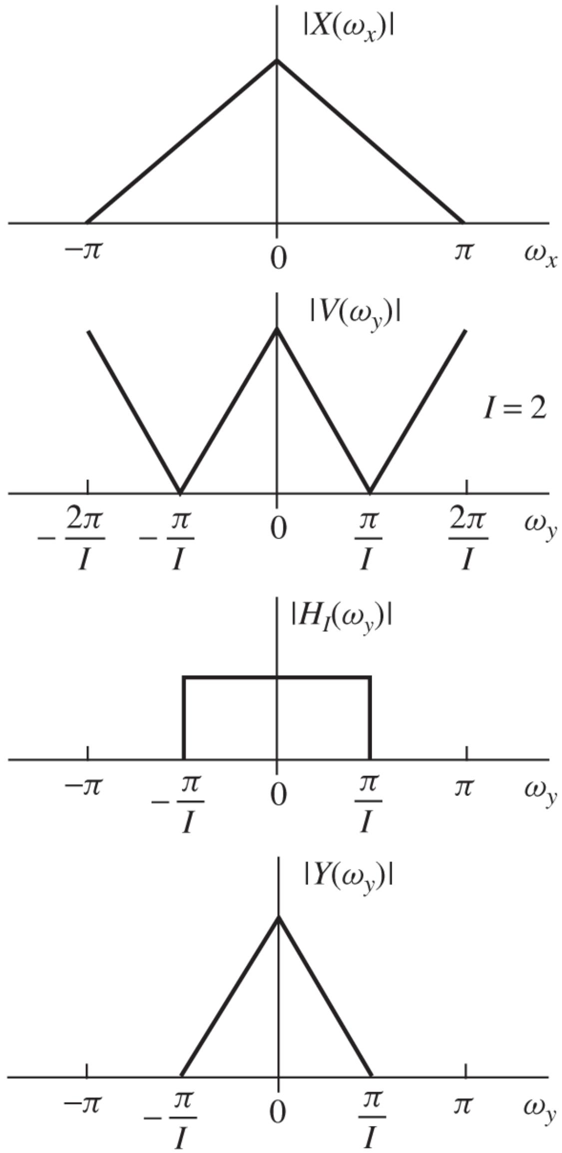

Figure 11.3.1 Spectra of $x(n)$ and $v(n)$ where $V(\omega_y) = X(\omega_y l)$.